A STUDY OF THE THEORY AND MEASUREMENT OF PLASMA DIFFUSION*

by

Gerald A. Otteni

Technical Report No. A-35

April, 1965

Supported in part by the National Aeronautics and Space Administration.

---

*This investigation was submitted in partial fulfillment of the requirements for the Degree of Master of Science.

ERRATA

Page	Line	Equation	Correction
6	20		Change (2) to (1-2)
8	17		Change quantitites to quantities
17	5		Change satisifes to satisfies
21	9	(2-20)	Equation should read: $\mathcal{H}(\omega,s) = \dfrac{\phi_{nm}(\omega)}{[(s-\nu)+D_{xx}(\frac{n\pi}{a})^2 + D_{yy}(\frac{m\pi}{b})^2 + D_{zz}\omega^2]}$
22	9		.... be a delta function ....
30	14		Change inital to initial
48	10	(A-5)	Note that $\underline{B} = B_o \underline{a}_r$ is a fictitious field.
49	14		Change expasnion to expansion
50	1		Change Fig. 7 to Fig. 16
51	11	(A-9)	The coefficient a_{nm} has been deleted in the series for $n(r,\phi,z,t)$.
54	2		Change Fig. 8 to Fig. 17.
	3		.... already been ....
	3		Change page 13 to page 27.
56	3	(A-23)	Equation (A-23) should read: $\mathcal{R} = \dfrac{K(\omega)}{[(s-\nu)+D_{zz}(\frac{m\pi}{c})^2+D_{rr}\omega^2]}$
	10	(A-25)	Change $\delta\frac{(r-r')}{r}$ to $\frac{\delta(r-r')}{r}$.
		(A-26)	Change $\delta\frac{(r-r')}{r}$ to $\frac{\delta(r-r')}{r}$.
57	10	(A-30)	∞ has been deleted as limit of summation
	10	(A-30)	Change $\exp[(\nu-D_{zz}(m\pi/c\ ^2)t]$ to $\exp\left\{[\nu-D_{zz}(m\pi/c)^2]t\right\}$.
	13	(A-31)	Change z to z' .
58	1	(A-32)	Add closing bracket to G( ) .
	2	(A-32)	$\ldots \cos n\phi\,[\frac{1}{(2D_{rr}t)}]\exp[\quad]$
	3	(A-32)	$\ldots[\frac{1}{2(\pi D_{zz}t)^{1/2}}]\exp\left\{-[\frac{(z-z')}{2(D_{zz}t)^{1/2}}]^2\right\}$

Page	Line	Equation	Correction
60			Change polar angle and polar angle vector to θ and $\underline{a}_\theta$ in Fig. 19
60	5		Note that $\underline{B} = B_o \underline{a}_r$ is a fictitious field.
64	1		Change characteristic normalized impedance to normalized characteristic impedance.

ABSTRACT

15390

A basic introduction to the theory of plasma diffusion is given with references to previous papers in this area. Emphasis is placed on characterizing the diffusion by dyadic diffusion coefficients so that diffusion in the presence of magnetic fields can be included. In the theory, only attachment, ionization and diffusion mechanisms are assumed to exist in the plasma. Solutions are given for plasma distributions in rectangular and cylindrical geometries. An experiment is described which allows the plasma properties to be measured by reflection of a dominant TE_{10} rectangular waveguide mode from a decaying plasma distribution which partially fills the waveguide. Preliminary results in the absence of a magnetic field indicate the presence of a number of resonances in the plasma slab.

author

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professors Robert E. Collin and Osman K. Mawardi for guidance and discussions of this work. This work was supported in part by the National Aeronautics and Space Administration. A special note of thanks is due Mrs. Martha S. Dybas for typing this manuscript.

TABLE OF CONTENTS

CHAPTER I

COLLISIONAL DIFFUSION

1.1 Introduction

The study of diffusion of electrons and ions in an ionized gas is certainly not a relatively new topic. Early studies, based on normal binary electron-atom collision theory, were conducted by J. S. Townsend as early as 1912.[1] Townsend developed the well known collisional diffusion coefficients for a weakly ionized gas in the presence of a uniform axial magnetic field. For singly ionized atoms,

$$D_{i\perp} = D_i/(1 + \omega_i^2 \tau_i^2) \ , \ \ D_{e\perp} = D_e/(1 + \omega_e^2 \tau_e^2) \quad , \qquad (1\text{-}1)$$

where

$$\omega_i = eB/m_i \ ; \quad \omega_e = eB/m_e \ ;$$

$D_{i,e}$ is the diffusion coefficient of ions or electrons in the absence of the magnetic field, e is the electronic charge, $m_{i,e}$ is the ion or electron mass and $\tau_{i,e}$ is the binary collision interval between ion or electron collisions with neutrals.

Further study of plasma diffusion was made by Schottky, who considered the case of equal electron and ion currents, ambipolar diffusion, for the positive column.[2]

Since this early work, many authors have treated the subject of plasma diffusion with and without magnetic fields. Within the last 25 years and especially within the last 10 years, a great effort has been made in this direction to understand the processes involved. Impetus has been given to this study, perhaps, by the desire to confine plasmas with magnetic fields.

A number of good reviews of theory and experiment exist, such as those by Golant,[3] Hoh,[4] and Oskam.[5] Extensive references are given in these papers.

1.2 Collisional Diffusion — Boltzmann Approach

Phase Space - If the position of a particle is known to occupy a certain region of real space, that is, to lie within a range of coordinates $x + dx$, $y + dy$, $z + dz$ and also has associated with it a velocity known to be in the range $v_x + dv_x$, $v_y + dv_y$, $v_z + dv_z$, the particle can be described in a 6-dimensional phase space by specifying its coordinate position. If a group of particles occupy this phase space, the complete specification of the system is to specify the number of particles occupying each elemental volume

$d^6\tau = dxdydzdv_x dv_y dv_z$ of phase space at each instant of time. This specification of particles must necessarily be discrete, by definition, but if the number of particles in phase space is large and the elemental volume $d^6\tau$ small, a good approximation is obtained by specifying the number of particles in the space by a continuous distribution function, $f(\underline{r},\underline{v},t)$, rather than its discrete form.

Boltzman Equation - If the phase space point of view is taken for a plasma and all possible ways of changing the electron (or ion) distribution function $f(\underline{r},\underline{v},t)$ are considered, the following equation, the Boltzmann equation, results:

$$\partial f/\partial t + \sum_{j=1,2,3} \partial f/\partial x_j \, dx_j/dt + \sum_{j=1,2,3} \partial f/\partial v_j dv_j/dt$$

$$= (\partial f/\partial t)_{\text{due to collisions}} \qquad (1\text{-}2)$$

where $j=1,2,3$ correspond to the coordinates x,y,z. A detailed discussion of the Boltzmann equation is given by Chapman and Cowling.[6]

The procedure to be followed in obtaining the continuity equation and diffusion coefficient expressions parallels that of Spitzer,[7] Delcroix[8] and Golant[3]. When (1-2) is integrated over all velocity space, the continuity equation is obtained

$$\partial n/\partial t + \nabla.(n\underline{\bar{v}}) = (\partial n/\partial t)_{coll.} \qquad (1\text{-}3)$$

where

$$n = \iiint_{-\infty}^{\infty} f(\underline{r},\underline{v},t)\ dv_x dv_y dv_z$$

is the electron density distribution function $n(\underline{r},t)$ and $\underline{\bar{v}}$ is the mean electron velocity of the system of electrons.

There are a number of ways in which the number of electrons in a plasma may be changed by collisions:

1) ionization collisions
2) recombination collisions
3) attachment collisions.

Ionization Collisions - The dominant ionizing mechanism in a weakly ionized gas is normally the electron-atom collision. Electrons in the gas are accelerated by external electric fields, either steady or alternating, or by some other means until they have enough energy to ionize atoms of the gas. A collision of an electron of sufficient energy to ionize an atom with an atom of the gas may then ionize the atom. The law of mass action gives the time rate of increase of electrons by this process of collision. The relationship between density and time rate of increase is written

$$\partial n/\partial t \;\alpha\; n_o n = \nu_i n \qquad (1\text{-}4)$$

where ν_i is the electron-atom ionization collision frequency, n_o is the neutral atom density function.

Recombination Collisions - Recombination may be quite complicated for some ionized gases where two or three reactions are necessary before an electron has the correct energy to recombine with an ion to form the atom. However, in many gases, the dominant process involves capture of an electron by a singly ionized atom. The rate of recombination for this process proceeds as

$$\partial n/\partial t = -k_o n_i n \qquad (1\text{-}5)$$

where n_i is the ion distribution function. Usually, it is assumed that the plasma is neutral $(n \doteq n_i)$. For this case,

$$\partial n/\partial t = -kn^2 \qquad (1\text{-}6)$$

is the equation for electron recombination loss; k is the proportionality constant which is called the recombination coefficient.

<u>Attachment</u> - In non-noble gases there exists an affinity of an atom for another electron so that free electrons can be attached to atoms to produce negative ions. The relationship between the rate of attachment and the electron density can be written

$$\partial n/\partial t \quad \alpha \quad n_o n = -\nu_a n \qquad (1\text{-}7)$$

where ν_a is the attachment coefficient, n_o the neutral density distribution function.

Oskam[5] discusses volume and surface recombination and attachment of electrons in a plasma volume.

For the processes that are considered here, the continuity equation, (1-3), becomes

$$\nabla.(n\underline{\bar{v}}) + (\nu_a - \nu_i)n + kn^2 = -\partial n/\partial t \qquad (1\text{-}8)$$

where $(\nu_i - \nu_a)n - kn^2$ has been substituted for $(\partial n/\partial t)_{coll.}$. Equation (1-8) can be expressed in a more convenient form if the velocity can be expressed as a function of the density n . If the momentum equation for the ionized gas is considered and all possible ways of changing the momentum of the electron is accounted for, an expression for $n\underline{\bar{v}}$ in terms of n can be obtained. The momentum equation is obtained by multiplying the Boltzmann equation, (2), by the velocity $\underline{v}$ and again integrating over velocity space to obtain the average space momentum equation. Following Spitzer[8] and Delcroix[7], the momentum equation can be written in the following form:

$$nm(\partial\underline{\bar{v}}/\partial t + \underline{\bar{v}} \cdot \nabla\underline{\bar{v}}) = -ne(\underline{E} + \underline{\bar{v}} \times \underline{B}) - \nabla \cdot \bar{\psi} - nm\nabla\phi$$
$$+ \underline{P} + (\nu_i - \nu_a)nm\underline{\bar{v}} - kn^2 m\underline{\bar{v}} \qquad (1\text{-}9)$$

where m is the electron mass, e is the electronic charge, $\underline{B}$ is the magnetic field, $\bar{\psi}$ is the stress tensor, ϕ is the gravitational potential, $\underline{P}$ is the net average momentum gain due to elastic collisions with other particles. The continuity equation (1-8) has been used to reduce (1-9) to its present form. The result is different from the results obtained by Spitzer and Delcroix since they considered no changes in the number of particles due to collisions.

When the pressure is a scalar quantity, $-\nabla \cdot \bar{\psi}$ is written $-\nabla p$, where p is the pressure and for a Maxwellian velocity distribution is equal to nKT_e where T_e is the electron temperature, K is Boltzmann's constant. The pressure here is the electron partial pressure $p_e = nKT_e$ because the momentum equation is written specifically for electrons.

For a weakly ionized gas, the momentum term $\underline{P}$ can be approximated by $-nm_e \nu_{en} \overline{\underline{v}}$, where ν_{en} is the electron-neutral collision frequency. It is assumed here that the electron-ion collisions are much less probable than the electron-neutral collisions. In addition, it is assumed that the directed velocity, $\overline{\underline{v}}$, and its time rate of change are small in comparison to the electron-neutral collision term and the random electron velocity respectively. With these approximations in mind, the left side of (1-9) is neglected and (1-9) reduces to

$$e(\underline{E} + \overline{\underline{v}} \times \underline{B}) + KT_e \nabla n/n + [(\nu_a - \nu_i) + kn + \nu_{en}]m\overline{\underline{v}} = 0 \,. \qquad (1\text{-}10)$$

It is also assumed that T_e is constant throughout the plasma volume.

The solution for $\overline{\underline{v}}$ can be obtained by taking the projection of equation (1-10) in the direction of $\overline{\underline{v}}$; for example, $\overline{v}_x$ is found by taking the scalar product of the vector equation (1-10) with the unit vector $\underline{a}_x$. The general result can be written in a compact form if tensor quantitites are introduced. The result for $\overline{\underline{v}}$ is given as

$$\overline{\underline{v}} = - \overline{D}.[\nabla n/n] + \overline{\mu}. \underline{E} \qquad (1\text{-}11)$$

where $\overline{D}$ is defined to be the diffusion tensor and $\overline{\mu}$ is the tensor mobility.

The continuity equation, (1-3) can now be written in a different form with the aid of (1-11):

$$\nabla.(\overline{D}.\nabla n) - \nabla.(\overline{\mu}.\underline{E}n) + (\nu_i - \nu_a)n - kn^2 = \partial n/\partial t . \qquad (1\text{-}12)$$

The diffusion coefficient as defined by eqs. (1-10) and (1-11) is necessarily space and time dependent because of the recombination term kn in equation (1-10) . Normally, however, the electron-neutral collision frequency is dominant over the recombination, attachment and ionization terms so that these coefficients may normally be neglected in comparison to the electron-neutral collision term. Most calculations of the diffusion coefficient and mobility tensors are made under this assumption.[9]

For the case in which the total average electric field in the plasma vanishes, the continuity equation reduces to a form involving only the density function n :

$$\nabla.(\overline{D}.\nabla n) + (\nu_i - \nu_a)n - kn^2 = \partial n/\partial t . \qquad (1\text{-}13)$$

Another condition of plasma diffusion exists such that the electric field can be expressed in terms of n and the continuity equation again reduces to the form given in (1-13) .

1.3 Ambipolar Diffusion

A very special diffusion condition may exist in a plasma such that the average electron current, $n_e \, \overline{\underline{v}}_e$, is just equal to the ion current, $n_i \, \overline{\underline{v}}_i$, so that the net current in the plasma is zero. This condition exists in a steady-state plasma volume where the walls are insulated so that the wall conditions prescribe the equality of currents throughout the plasma volume. Physically, the equality of currents indicates that the charge separation of electrons and ions couples the electron and ion flows by means of the electric field which the separation produces. This ambipolar process can be described by regular diffusion theory if the diffusion coefficients are changed to the ambipolar diffusion coefficients. In order to obtain the diffusion expression for ambipolar diffusion, it is necessary to express the electric field in terms of the electron density n .

Equations (1-10) and (1-11), as they are written, are just as valid for ions as electrons. Using equation (1-11) and equating the electron and ion currents in a plasma gives:

$$n_e \, \overline{\underline{v}}_e = n_i \, \overline{\underline{v}}_i = -\overline{D}_e \cdot \nabla n_e + n_e \, \overline{\mu}_e \cdot \underline{E}$$

$$= - \overline{D}_i \cdot \nabla n_i + n_i \overline{\mu}_i \cdot \underline{E} \qquad (1\text{-}14)$$

where the subscripts e, i refer to the electron and ion properties respectively. The equation may be solved for the electric field, $\underline{E}$, with the result:

$$\underline{E} = (n_i \overline{\mu}_i - n_e \overline{\mu}_e)^{-1} \cdot (\overline{D}_i \cdot \nabla n_i - \overline{D}_e \cdot \nabla n_e) \qquad (1\text{-}15)$$

where $(\ )^{-1}$ here indicates the inverse matrix operation. It is also assumed here, for the case of perfect ambipolar diffusion, that $n_e \doteq n_i = n$ so that the expression for the electric field reduces in its final form to

$$\underline{E}_a = (\overline{\mu}_i - \overline{\mu}_e)^{-1} \cdot (\overline{D}_i - \overline{D}_e) \cdot \nabla n / n \ . \qquad (1\text{-}16)$$

If this expression for the electric field is substituted into the continuity equation (1-12) , the continuity equation reduces to the expression

$$\nabla \cdot (\overline{D}_a \cdot \nabla n) + (\nu_i - \nu_a) n - kn^2 = \partial n / \partial t \qquad (1\text{-}17)$$

where $\overline{D}_a = \overline{D}_e - \overline{\mu}_e \cdot (\overline{\mu}_i - \overline{\mu}_e)^{-1} \cdot (\overline{D}_i - \overline{D}_e)$.

1. 4 Measurements

In order to measure the properties of a plasma such as diffusion, recombination and ionization, it is necessary to know the density function as a function of time. The exact solution of the continuity equation (1-13), (1-17), has not been obtained except in the one dimensional problem because of the non-linearity in the density function n. It is therefore difficult to assess to a measurement of n the effect of recombination. If the recombination term is neglected, the exact solution of the continuity equation is possible in most coordinate system. Before any simplifying assumptions are made, however, a consideration of the recombination process is necessary.

A numerical calculation of the density function for processes in which diffusion and recombination exist simultaneously has been made by Gray and Kerr.[10] The validity of many microwave measurements of recombination is considered in their paper.

CHAPTER II

SOLUTIONS OF THE DIFFUSION PROBLEM

In a plasma diffusion problem, the equation describing the diffusion process which includes ionization and attachment but neglects recombination is written as

$$\nabla.(\overline{D}.\ \nabla n) + (\nu_i - \nu_a)n = \partial n/\partial t \qquad (2\text{-}1)$$

where $\overline{D}$ is the diffusion tensor, n is the electron density distribution function, ν_i is the ionization collision frequency and ν_a is the attachment collision frequency. In equation (2-1) the electron flow term $\underline{\Gamma}_e = n\ \underline{\overline{v}}$ is written as $-\overline{D}.\ \nabla n$. It is assumed here that $\overline{D}$, ν_i and ν_a are not functions of position and no temperature gradients are considered. If temperature gradients cause $\overline{D}$ to be space dependent, the additional diffusion term can be added as $n\partial/\partial x_k\ D_{jk} = n\nabla\ .\ \overline{D}^T$ where $\overline{D}^T$ indicates the transpose of $\overline{D}$.[11]

2.1 Diffusion in a Rectangular Geometry

For the rectangular case, (2-1) reduces to

$$\nabla \cdot [(D_{xx}\partial n/\partial x + D_{xy}\partial n/\partial y + D_{xz}\partial n/\partial z)\ \underline{a}_x$$
$$+ (D_{yx}\partial n/\partial x + D_{yy}\partial n/\partial y + D_{yz}\partial n/\partial z)\ \underline{a}_y$$
$$+ (D_{zx}\partial n/\partial x + D_{zy}\partial n/\partial y + D_{zz}\partial n/\partial z)\ \underline{a}_z]$$
$$+ (\nu_i - \nu_a)n = \partial n/\partial t \ . \qquad (2\text{-}2)$$

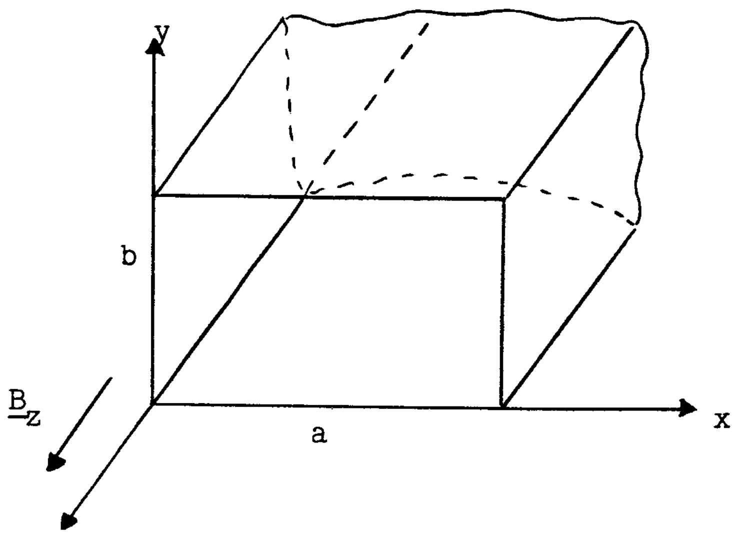


Fig. 1

If no magnetic field exists, there is no net coupling between a diffusive flow in one direction to any other direction, so the diffusion tensor is

$$\overline{D} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix} \qquad (2\text{-}3)$$

where D is the free space diffusion coefficient. The continuity equation for the plasma then reduces to

$$D(\partial^2 n/\partial x^2 + \partial^2 n/\partial y^2 + \partial^2 n/\partial z^2) + (\nu_i - \nu_a)n$$

$$= \partial n/\partial t \ . \qquad (2\text{-}4)$$

If, on the other hand, a uniform steady-state magnetic field is applied, the diffusion tensor now takes on the general form

$$\bar{D} = \begin{vmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{vmatrix} . \qquad (2\text{-}5)$$

Suppose for argument that the magnetic field is $+z$ directed as in Fig. 1 (page 14) . Then, a unit gradient of n in the $\underline{a}_x$ direction causes a flow D_{yx} of electrons in the $-\underline{a}_y$ direction due to the $-e\ \underline{v} \times \underline{B}$ force on the electrons. A unit gradient of n in the $\underline{a}_y$ direction causes an electron flow in the $\underline{a}_x$ direction equal to D_{xy} . Since the space is homogeneous, the two flows must be equal in magnitude and therefore $D_{yx} = -D_{xy}$. A more general analysis of microscopic irreversibility is treated by H. B. G. Casmir.[12] For this geometry and magnetic field orientation it is also found by similar argument that and

$$D_{xz} = D_{zx} = D_{yz} = D_{zy} = 0$$

$$D_{xx} = D_{yy} = D_{\perp} \quad ; \; D_{zz} = D_{11} \; .$$

So

$$\bar{D} = \begin{vmatrix} D_{\perp} & D_{xy} & 0 \\ -D_{xy} & D_{\perp} & 0 \\ 0 & 0 & D_{11} \end{vmatrix} \; . \tag{2-6}$$

If the analysis is extended by superposition of magnetic fields to obtain a general orientation of the magnetic field, $\underline{B} = B_1 \, \underline{a}_x + B_2 \, \underline{a}_y + B_3 \, \underline{a}_z$, the general dyadic form of the diffusion tensor becomes

$$\bar{D} = \begin{vmatrix} D_{xx} & D_{xy} & D_{xz} \\ -D_{xy} & D_{yy} & D_{yz} \\ -D_{xz} & -D_{yz} & D_{zz} \end{vmatrix} \; . \tag{2-7}$$

The general tensor can be obtained by adding the skew-symmetric parts of the diffusion tensors obtained when the three components of magnetic field are assumed to act individually and adding the corresponding diagonal parts essentially as series conductances. In general,

$$\bar{D}_{ss} = \bar{D}_{1ss} + \bar{D}_{2ss} + \bar{D}_{3ss} = \text{skew-symmetric part of } \bar{D}$$

$$D_{kk} = \frac{1}{\frac{1}{D_{kk_1}} + \frac{1}{D_{kk_2}} + \frac{1}{D_{kk_3}} - \frac{2}{D}} \qquad (2\text{-}8)$$

$k = x, y, z$

where the subscripts 1, 2, 3 correspond to B_1, B_2, and B_3. This formula for obtaining the general tensor assumes that the diffusion satisifes normal diffusion theory, that is $D_\perp = \frac{D}{1+\omega_c^2\tau^2}$ where D is the free space diffusion constant, $\frac{eB}{m} = \omega_c$ is the electron cyclotron frequency, and τ is the mean time between collisions of the electron and neutral particle. If a particular component of the magnetic field is zero, say B_3, then $D_{xx_3} = D_{yy_3} = D_{zz_3} = D$.

The general diffusion tensor then is of the form $\bar{D}=\bar{D}_d + \bar{D}_{ss}$ where $\bar{D}_d$ is a diagonal tensor and $\bar{D}_{ss}$ is the skew-symmetric part of $\bar{D}$. The significance associated with this form of the diffusion tensor is shown when the expression $\nabla.(\bar{D}.\nabla n)$ is expanded. For the diffusion tensor in this form, it is easily seen that $\nabla.(\bar{D}.\nabla n)$ in rectangular coordinates reduces to

$$D_{xx}\, \partial^2 n/\partial x^2 + D_{yy}\partial^2 n/\partial y^2 + D_{zz}\partial^2 n/\partial z^2 \,. \qquad (2.9)$$

Therefore, electron losses due to diffusion depend only on the diagonal elements of the diffusion tensor or the diffusion coefficients along the coordinate axes in a rectangular coordinate system and there are no mixed partial derivatives in the expansion of the continuity equation.

The reduction of the diffusion equation to this form can also be shown easily by considering the del operator (∇) as a linear operator for matrix multiplication. For rectangular coordinates, the divergence operator is equal to the gradient operator so the matrix equation can be written in the following form by defining the del operator as $\nabla \equiv X$ operator.

$$\nabla.(\overline{D}.\nabla n) = (\nabla.\overline{D}.\nabla)n \equiv (X^T\overline{D}X)n = [X^T(\overline{D}_d + \overline{D}_{ss})\ X]n$$

$$= (X^T\overline{D}_d\ X + X^T\overline{D}_{ss}\ X)n$$, but since $\overline{D}_{ss}$ is skew-symmetric, it has the property that $\overline{D}^T_{ss} = -\overline{D}_{ss}$.

Therefore,

$$(X^T\overline{D}_{ss}\ X)^T = (X^T\overline{D}_{ss}\ X) = -X^T\overline{D}_{ss}\ X\ .$$

The matrix $(X^T\overline{D}_{ss}\ X)$ is a square matrix of order 1, so it must be equal to its transpose. Thus it is shown that

$$X^T\overline{D}_{ss}X = -X^T\overline{D}_{ss}\ X = 0\ .$$

Then $(\nabla.[\overline{D}.\nabla n])$ always reduces to $(\nabla.[\overline{D}_d.\nabla n])$, where $\overline{D}_d$ is the main diagonal portion of $\overline{D}$. It is therefore concluded that any diffusion problem expressed in rectangular coordinates with a uniform steady-state magnetic field of any orientation can be expressed in terms of the principal diffusion coefficients only and that the cross terms in the expansion of the continuity equation do not enter into the problem. The equation to be solved in considering diffusion with a rectangular coordinate system is then

$$D_{xx}\partial^2 n/\partial x^2 + D_{yy}\ \partial^2 n/\partial y^2 + D_{zz}\ \partial^2 n/\partial z^2$$
$$+ (\nu_i - \nu_a)n = \partial n/\partial t \ . \qquad (2\text{-}10)$$

2.2 Solutions of Diffusion in a Rectangular Geometry

In order to obtain a unique solution for the diffusion problem, it will be assumed that all electrons that reach the walls of the rectangular container are recombined or lost from the plasma volume, that is, $n_{wall} = 0$. It is also assumed that $n = n(x,y,z,t)$ is a product solution of the form

$$n(x,y,z,t) = f(x)\ g(y)\ h(z,t)$$
$$= \sum_{n=0,m=0}^{\infty} a_{nm} \sin n\pi x/a \sin m\pi y/b\ h(z,t) \ . \qquad (2\text{-}11)$$

Applying the Laplace transform

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{2-12}$$

to (2-10) gives

$$D_{xx}\partial^2N/\partial x^2 + D_{yy}\partial^2N/\partial y^2 + D_{zz}\ \partial^2N/\partial z^2 + \nu N = sN - N_o \tag{2-13}$$

where N is the Laplace transform of $n(x,y,z,t)$, $\nu = \nu_i - \nu_a$ and N_o is the initial density function $n(x,y,z,0)$.

From (2-11) and (2-12) ,

$$N(x,y,z,s) = f(x)\ g(y)\ h(z,s) = \sum_{n=0,m=0}^{\infty} a_{nm} \sin n\pi x/a \sin m\pi y/b\ h(z,s) \tag{2-14}$$

where $h(z,s)$ is the Laplace transform of $h(z,t)$. It is also assumed that

$$N_o = \sum_{n=o,m=o}^{\infty} b_{nm} \sin n\pi x/a \sin m\pi y/b\ p(z)\ . \tag{2-15}$$

Substituting (2-13) and (2-14) in (2-15) and equating terms gives

$$[s - \nu + D_{xx}(n\pi/a)^2 + D_{yy}(m\pi/b)^2]\ h(z,s) + D_{zz}\ \partial^2h/\partial z^2 = c_{nm}\ p(z) \tag{2-16}$$

where

$$c_{nm} = b_{nm}/a_{nm}\ . \tag{2-17}$$

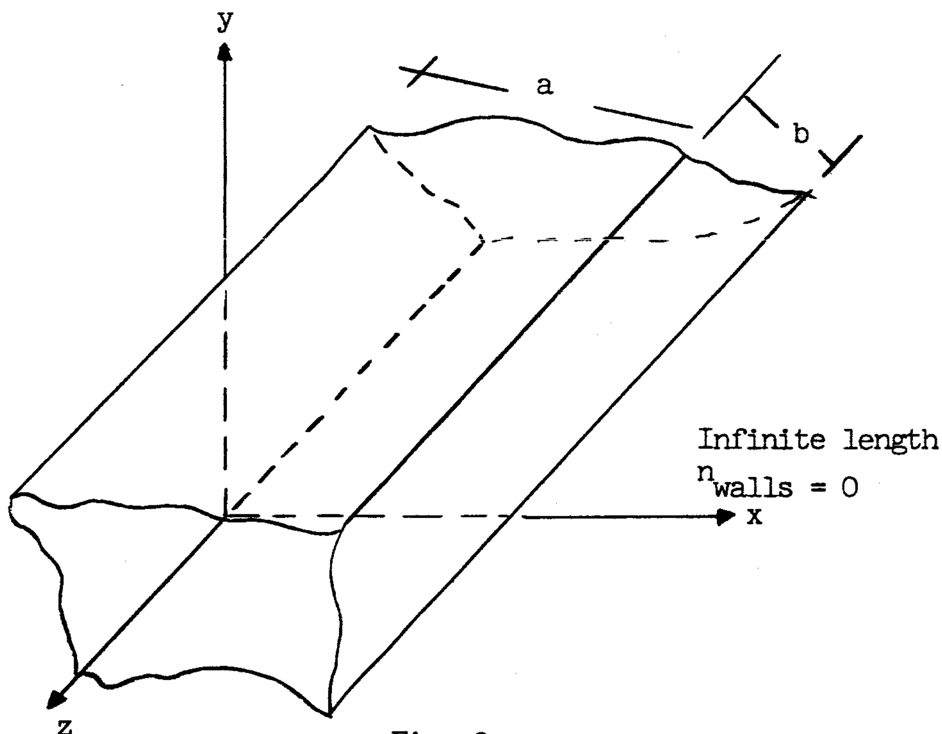


Fig. 2

Green's function solution for infinite length rectangular cylinder.

For a plasma container of infinite length in the z direction, equation (2-16) can be transformed using the Fourier transform

$$\mathscr{H}(\omega, s) = 1/2\,\pi \int_{-\infty}^{\infty} h(z,s)\, e^{-j\omega z}\, dz \qquad (2\text{-}18)$$

giving

$$[(s - \nu) + D_{xx}(n\pi/a)^2 + D_{yy}(m\pi/b)^2 + D_{zz}\omega^2]\mathscr{H} = \phi_{nm}(\omega) \qquad (2\text{-}19)$$

where $\phi_{nm}(\omega)$ is the Fourier transform of $c_{nm}\, p(z)$.

Thus,

$$\mathscr{H}(\omega, s) = \phi_{nm}(\omega) \Big/ (s-\nu) + D_{xx}(n\pi/a)^2 + D_{yy}(m\pi/b)^2 + D_{zz}\omega^2 \, . \qquad (2\text{-}20)$$

The inverse Laplace transform of (2-20) gives

$$H(\omega,t) = \phi_{nm}(\omega)\ \exp[\nu - D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2]t\ \exp[-D_{zz}\omega^2 t]\ . \tag{2-21}$$

Applying the inverse Fourier transform to obtain the z variable yields

$$h(z,t) = \exp[\nu - D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2]t$$

$$\int_{-\infty}^{\infty} \phi_{nm}\ \exp[-D_{zz}\omega^2 t]\ \exp[j\omega z]d\omega\ . \tag{2-22}$$

In order to obtain the Green's function solution for $n(x,y,z,t)$, the initial distribution function $N_o = n(x,y,z,0)$ must be delta function

$$N_o = \delta(x-x')\delta(y-y')\delta(z-z')$$

and from (2-15)

$$N_o = \sum_{\substack{n=0\\m=0}}^{\infty} b_{nm} \sin(n\pi x/a) \sin(m\pi y/b)\ p(z)\ .$$

so that

$$p(z) = \delta(z-z') \tag{2-23}$$

and

$$\delta(x-x')\delta(y-y') = \sum_{\substack{n=0\\m=0}}^{\infty} b'_{nm} \sin(n\pi x/a) \sin(m\pi y/b)\ . \tag{2-24}$$

The Fourier coefficient b'_{nm} then is

$$b'_{nm} = (4/ab)\,\sin(n\pi x'/a)\,\sin(m\pi y'/b)\ . \qquad (2\text{-}25)$$

Also

$$\phi_{nm}(\omega) = (c_{nm}/2\pi)\int_{-\infty}^{\infty}\delta(z-z')\,\exp[-j\omega z]\,dz$$

$$= (c_{nm}/2\pi)\,\exp[-j\omega z']\ . \qquad (2\text{-}26)$$

Substituting this expression for $\phi_{nm}(\omega)$ into (2-22) gives

$$h(z,t) = \exp[\nu - D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2]t$$

$$\int_{-\infty}^{\infty}(c_{nm}/2\pi)\,\exp[-D_{zz}\omega^2 t]\,\exp[j\omega(z-z')]\,d\omega\ . \qquad (2\text{-}27)$$

By completing the square of the integrand and making a change of variables,

$$h(z,t) = \exp[\qquad]t[c_{nm}/2\pi(D_{zz}t)^{1/2}]\,\exp\{-[(z-z')/2(D_{zz}t)^{1/2}]^2\}$$

$$\int_{-\infty-j\alpha}^{\infty-j\alpha}\exp[-u^2]du\ . \qquad (2\text{-}28)$$

It is to be noted here that $h(z,t)$ is

$$c_{nm}\,\exp[-D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2]t$$

times the Green's function solution for the one dimensional solution to

$$D_{zz}\partial^2 n/\partial z^2 + \nu n = \partial n/\partial t \; ; \tag{2-29}$$

that is, the Green's function solution for (2-29) is written

$$G(z'|z,t) = [1/2(\pi D_{zz}t)^{1/2}] \exp[\nu t] \exp\{-[(z-z')/2(D_{zz}t)^{1/2}]^2\} . \tag{2-30}$$

Thus, from (2-11), (2-17), (2-28), the Green's function solution corresponding to the initial distribution $n(x,y,z,0) = \delta(x-x')\delta(y-y')\delta(z-z')$ and the geometry of Fig. 2 is

$$G(x|x', y|y', z|z', t) = G(r|r', t)$$

$$= \sum_{\substack{n=0\\m=0}}^{\infty} (4/ab)[1/2(\pi D_{zz}t)^{1/2}] \sin(n\pi x'/a) \sin(m\pi y'/b) \tag{2-31}$$

$$\sin(n\pi x/a) \sin(m\pi y/b) \exp[\nu - D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2]t$$

$$\exp\{-[(z-z')/2(D_{zz}t)^{1/2}]^2\} \qquad t>0 ,$$

$$= 0 , \; t<0 .$$

The solution $n(x,y,z,t)$ then for an arbitrary initial distribution function N_o satisfying $n_{walls} = 0$ and which has a Fourier series representation is given by superposition to be

$$n(x,y,z,t) = \int_0^a \int_0^b \int_{-\infty}^{\infty} N_o \, G(r|r', t) \, dx' \, dy' \, dz' . \tag{2-32}$$

As an example, consider the following initial distribution

$N_o = F(x)\,G(y)\,p(z)$ where $p(z)$ is defined as follows:

$$p(z) = A \,, \qquad -a_o \le z \le a_o$$
$$= 0 \,, \qquad |z| > a_o \,. \tag{2-33}$$

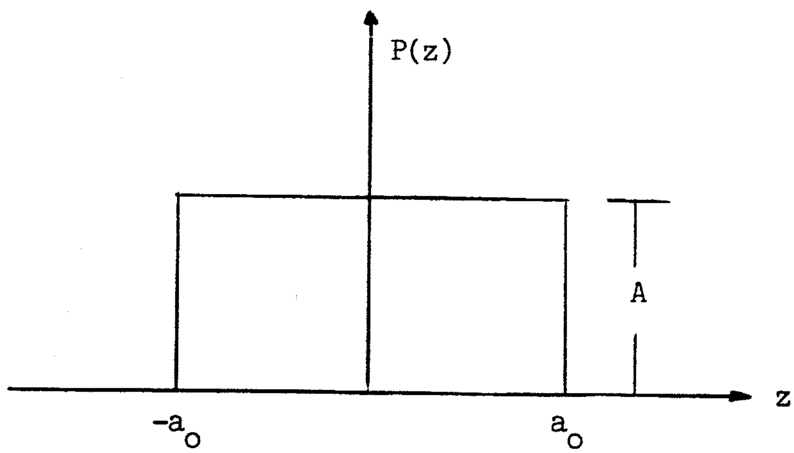


Fig. 3

Applying the superposition integral (2-32) gives

$$n(x,y,z,t) = \sum_{\substack{n=0\\m=0}}^{\infty} b_{nm} \sin(n\pi x/a)\,\sin(m\pi y/b)$$
$$(A/2)\,[\mathrm{erf}\{(a_o - z)/2(D_{zz}t)^{1/2}\}$$
$$+ \mathrm{erf}\{(a_o + z)/2(D_{zz}t)^{1/2}\}] \qquad t > 0 \tag{2-34}$$
$$= 0 \,, \quad t < 0$$

where b_{nm} is the Fourier coefficient of $F(x)\,G(y)$, $\mathrm{erf}(\eta)$ is the error function.

It is worthwhile to note that in the preceding development, it was assumed that $n_{walls} = 0$. If it had been assumed that the normal derivative of n at the wall was zero or a combination of these boundary conditions was applied, i.e., $\partial n/\partial x + \alpha n = 0$, the z dependent solution $h(z,t)$ would not have been altered, that is, the boundary conditions on the transverse walls do not affect the z dependence of the diffusion. The only difference in the solution would be the replacement of the sine terms by an appropriate combination of sine and cosine terms.

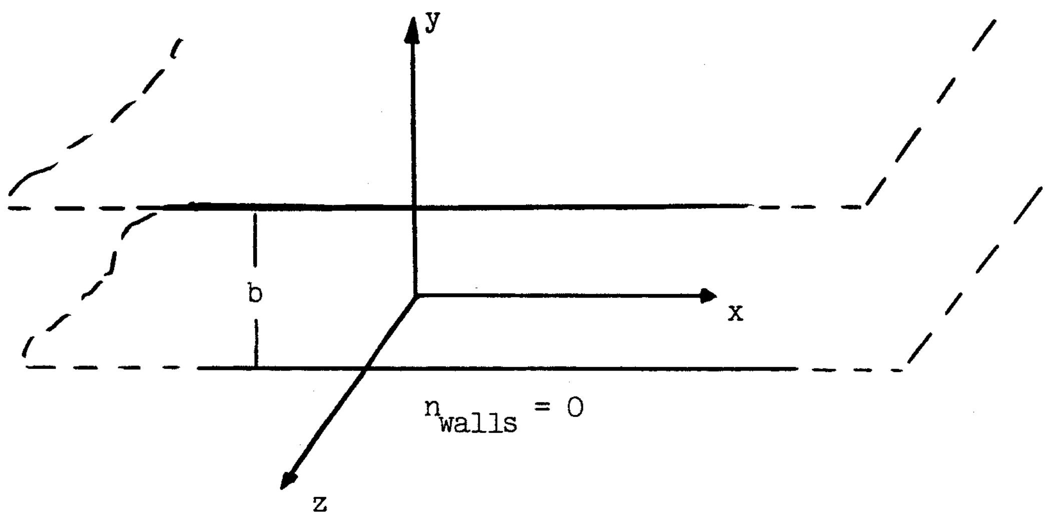


Fig. 4

Green's function solution for infinite parallel plane geometry.

The Green's function solution for the infinite parallel plane geometry can be obtained by a similar method used to obtain the solution for the infinite rectangular cylinder or by an analysis of (2-30) and (2-31) . The result is

$$G(x|x', y|y', z|z', t) =$$

$$\sum_{m=0}^{\infty} (2/b)\left[\frac{1}{2(\pi D_{xx}t)^{1/2}}\right]\left[\frac{1}{2(\pi D_{zz}t)^{1/2}}\right]$$

$$\sin(m\pi y'/b)\ \sin(m\pi y/b) \quad \exp[\nu - D_{yy}(m\pi/b)^2]t$$

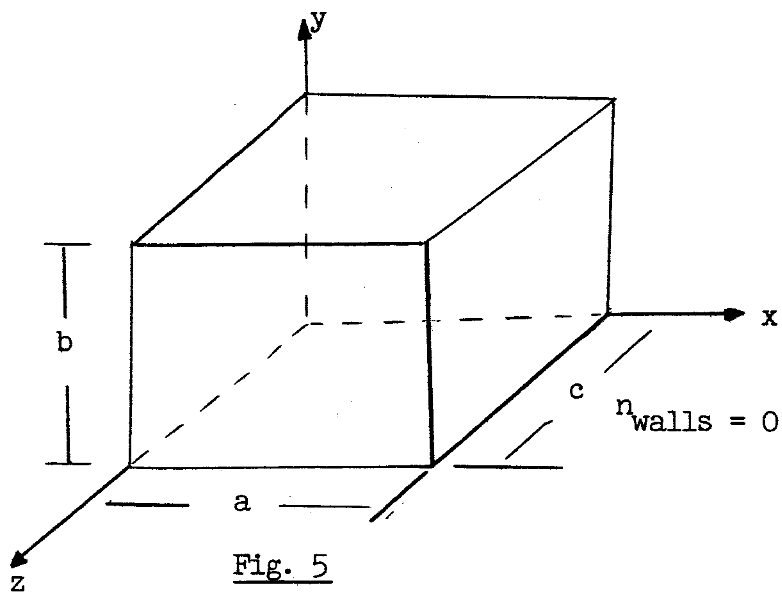
$$\exp\left[-\frac{(x-x')^2}{4D_{xx}t} - \frac{(z-z')^2}{4D_{zz}t}\right], \quad t > 0 . \tag{2-35}$$

<u>Green's function solution for an unbounded region</u> - For the unbounded region, the solution can be written again by analysis of (2-30) and (2-31) to be

$$G(r|r', t) = \left[\frac{1}{2(\pi D_{xx}t)^{1/2}}\right]\left[\frac{1}{2(\pi D_{yy}t)^{1/2}}\right]\left[\frac{1}{2(\pi D_{zz}t)^{1/2}}\right]$$

$$\exp[\nu t]\ \exp\left[-\frac{(x-x')^2}{4D_{xx}t} - \frac{(y-y')^2}{4D_{yy}t} - \frac{(z-z')^2}{4D_{zz}t}\right], \quad t > 0$$

$$= 0 , \quad t < 0 . \tag{2-36}$$



<u>Fig. 5</u>

Green's function solution for rectangular cavity of finite dimension - For the rectangular cavity the Green's function solution can be written

$$G(x|x', y|y', z|z', t) =$$

$$\sum_{\substack{n=0\\m=0\\l=0}}^{\infty} b'_{nml} \sin(n\pi x/a) \sin(m\pi y/b) \sin(l\pi z/c)$$

$$\exp[\nu - D_{xx}(n\pi/a)^2 - D_{yy}(m\pi/b)^2 - D_{zz}(l\pi/c)^2]t \qquad t>0 \tag{2-37}$$

$$= 0 \ , \ t<0 \ .$$

b'_{nml} is the Fourier coefficient expansion for the product of delta functions. If b_{nml} is the Fourier coefficient expansion of the initial density distribution function N_o then the solution obtained is actually $n(x,y,z,t)$ as is easily seen when the superposition integral for this problem is used. A similar statement can be made for all of the preceding solutions in rectangular coordinates.

For the delta function initial distribution,

$$b'_{nml} = (8/abc) \sin(n\pi x'/a) \sin(m\pi y'/b) \sin(l\pi z'/c) \ . \tag{2-38}$$

A discussion of diffusion for cylindrical and spherical geometries and solutions to the diffusion problem are given in Appendix A .

CHAPTER III

DESCRIPTION OF THE EXPERIMENT

3.1 Introduction

Numerous experiments have been performed to measure diffusion, recombination, attachment and ionization properties of gases by measuring the change in the Q of a cylindrical microwave cavity which contains a plasma.[13] Since this method usually uses a perturbation theory to account for the change in the microwave field, it is limited in its usefulness to low density plasmas or small dimension plasmas where the perturbation of the microwave field is small. The method can certainly be extended to higher density and larger dimension plasmas by a more exact solution of the microwave cavity field, but the field solutions in general are difficult to obtain and usually require computers to obtain numerical results.

With these ideas in mind, a method was designed to measure plasma diffusion where the field solutions were either easy to obtain or not extremely important to the theory and where the plasma system was not limited to small dimensions.

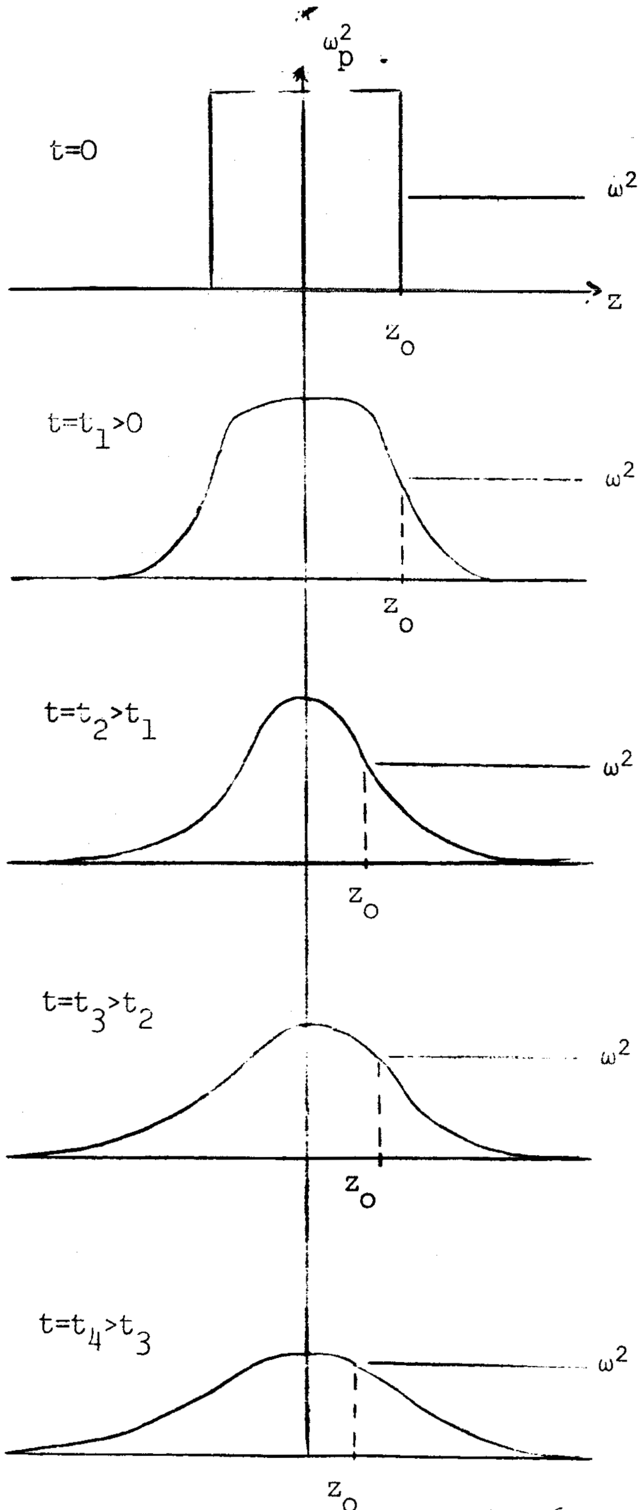
3.2 Experimental Bases

Geometry - A rectangular geometry was chosen for the experiment to simplify the field solutions and plasma density solutions as much as possible. Microwave measurement equipment in the 3-cm. range was available so the rectangular plasma container was chosen to fit inside a 3-cm. rectangular waveguide. This then allows the plasma properties to be obtained by a comparison between measurement of the reflection of the dominant TE_{10} waveguide mode from the plasma distribution and that calculated on a theoretical basis. Two possible regimes exist for this type of experiment:

1) the local plasma frequency exceeds the microwave probe signal frequency nearly everywhere in the plasma volume;

2) the local plasma frequency is everywhere less than the microwave probe signal frequency.

<u>Doppler Phase Shift Method</u> - For a plasma where the inital electron density is such that the plasma frequency ω_p is greater than the incident signal frequency ω almost everywhere, the method of Doppler shift is a useful means of measuring plasma properties. As an illustration of this method, consider the following one-dimensional problem where the plasma electron density proceeds in time as in Fig. 6 .



$\omega_p^2 = ne^2/m\varepsilon_o$

$\equiv$(plasma freq.)^2

n=electron density

e=electronic charge

ε_o=permittivity of free space

Fig. 6

A plane wave, $e^{jk_o z}$, incident from the right, is reflected by the plasma distribution essentially at the "classical turning point" or "reflection level" where the propagation constant in the plasma passes through zero value. For a collisionless Lorentz electron plasma model, the dielectric constant in the plasma is $\varepsilon = \varepsilon_o(1-\omega_p^2/\omega^2)$ so that the propagation constant in the plasma is $k_p = k_o(1-\omega_p^2/\omega^2)^{1/2}$. Essentially all of the incident plane wave is reflected from the classical turning point z_o where $\omega_p^2 = \omega^2$. Thus, as the plasma distribution diffuses along z, the turning point also changes and the reflection coefficient remains constant in magnitude but changes in phase. A measurement of the phase change then will provide a useful means of determining the plasma distribution when measurements are compared to theory. A necessary condition for this method requires that the plasma is reasonably thick so that the reflection comes essentially from the turning point z_o and any other turning point is not near z_o in terms of the electrical wavelength in the plasma.

This method can be applied to waveguide structures as long as only one dominant mode is excited and used for the measurement and the appropriate expression is used for the propagation constant in the guiding structure. A detailed discussion of this reflection problem is given by Budden[14] and Wait.[15]

Reflection Coefficient Method - When the local plasma frequency is everywhere less than the incident probe frequency, the plasma properties may be estimated by measuring the magnitude of the reflection coefficient of a decaying plasma and comparing the result with that obtained by calculating the reflection from the theoretical plasma distribution. This method for obtaining the plasma properties would be exact if the theory describing the plasma density were exact and the field solutions were obtained for this plasma distribution. One of the fundamental difficulties associated with this method as with any method of this type is the determination of correct boundary conditions and initial conditions for the plasma distribution.

3.3 System Operation

A 3-cm. pulsed magnetron ionizes the gas in the rectangular quartz tube located at the center of a waveguide cross. The plasma density thus created diffuses throughout the quartz tube and finally decays to zero. A 3-cm. klystron oscillator excites a TE_{10} waveguide mode to probe the plasma distribution and obtain information about the plasma properties. The reflected signal is then compared with the incident signal to measure the magnitude and phase of the reflected wave during the periods of ionization and decay of the plasma. A diagram of the ionization and measurement system is shown in Fig. 7 .

Isolation of the ionizing source and the probing signal is accomplished by the use of waveguide filters. Three band-rejection filters are tuned to the probe frequency and placed in the magnetron line to remove the portion of the magnetron spectrum that would interfere with the probe signal measurements. Additional band-pass filters, placed in the probe signal line and reflected line, pass the desired signal and reject the magnetron pulse.

At one end of the waveguide cross, in the probe line, a matched load was placed so that the probe line is matched to the klystron source when the plasma is absent. Also, a movable short was placed opposite the magnetron arm of the waveguide cross for the purpose of adjusting the impedance level of the system and matching the cross section to the probe line. A waveguide shorting switch was placed in the probe line to allow the level of complete reflection to be calibrated.

The quartz tube in the probe line extends a length of 4 inches on either side of the center of the magnetron line. The container is constructed of 1/16 inch quartz plate which has been fused to produce a rectangular cylinder that fits snugly in the 3-cm. waveguide. The ends are closed with 1/8 inch glass plate which has been epoxied in place. The tube is connected to the vacuum system by a length of 3/8 inch diameter glass tubing which passes through the matched load to a quick-coupling connector at

Plasma Ionization System

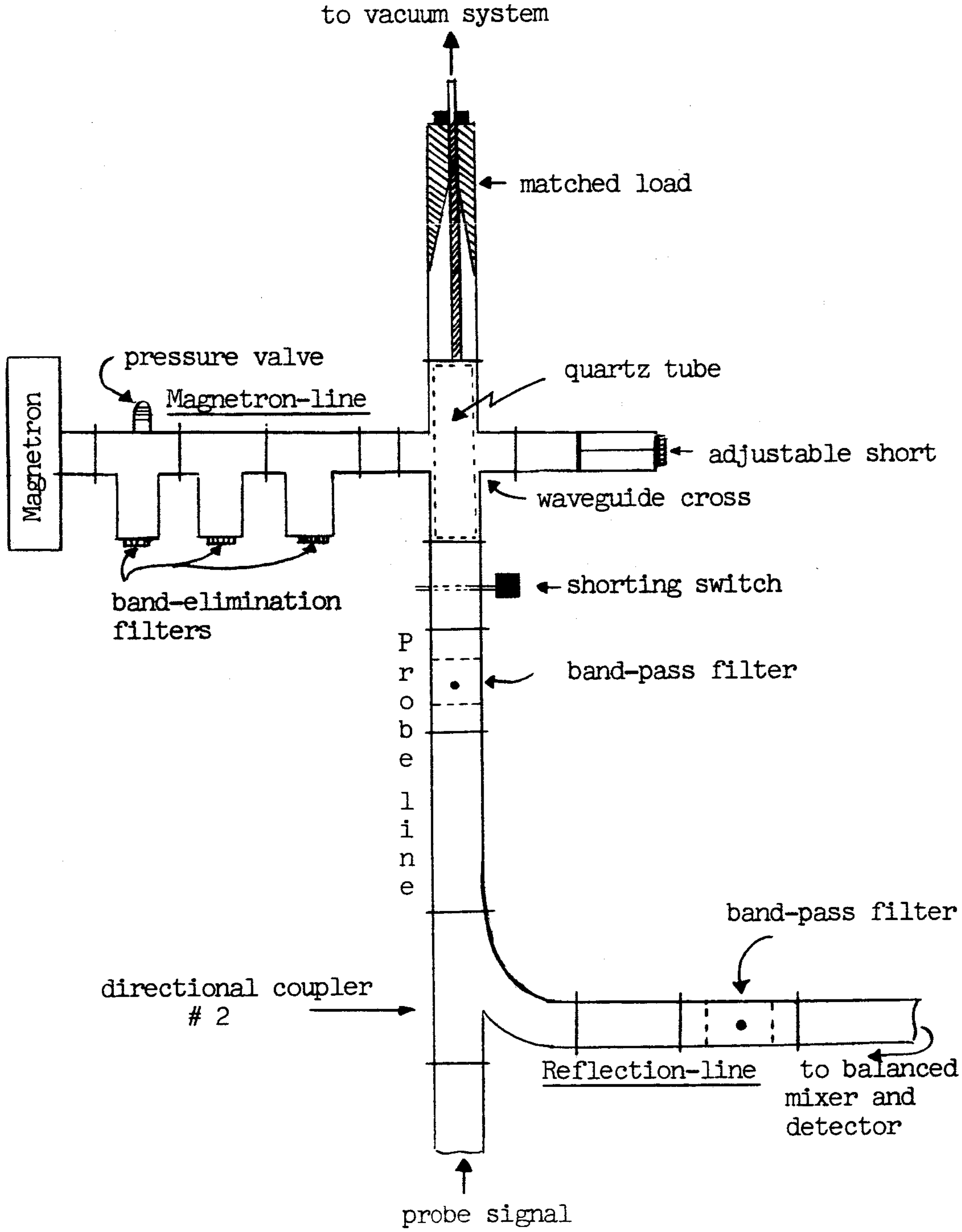


Fig. 7

the termination of the matched load. The magnetron line, cross, and matched load are pressurized at a pressure of 15 to 20 psi. to decrease arcing and breakdown outside the quartz tube.

A diagram of the phase measurement system is shown in Fig. 8 . The signal from the signal generator (klystron) of frequency f_o serves as the local oscillator for the balanced mixer. Part of the local oscillator signal is coupled by directional coupler # 1 to the helix of the traveling-wave tube where the signal is phase modulated at the sawtooth frequency f_1 to produce an amplified signal of frequency $f_o + f_1$. The signal passes through an attenuator and phase shifter, which are used to set the proper amplitude and phase shift of the probe signal, to the plasma system probe line.

A portion of the reflected wave is taken from directional coupler # 2 and sent to the signal input of the balanced mixer where the beat signal is obtained. The balanced mixer output is amplified by a differential amplifier and the resultant signal is applied to the signal input of the phase meter. At the phase meter, the phase of the signal from the differential amplifier is compared with the phase of the oscillator which synchronizes the sawtooth generator. The difference in phase between these two signals is the phase of the reflection coefficient.

Phase Measurement System

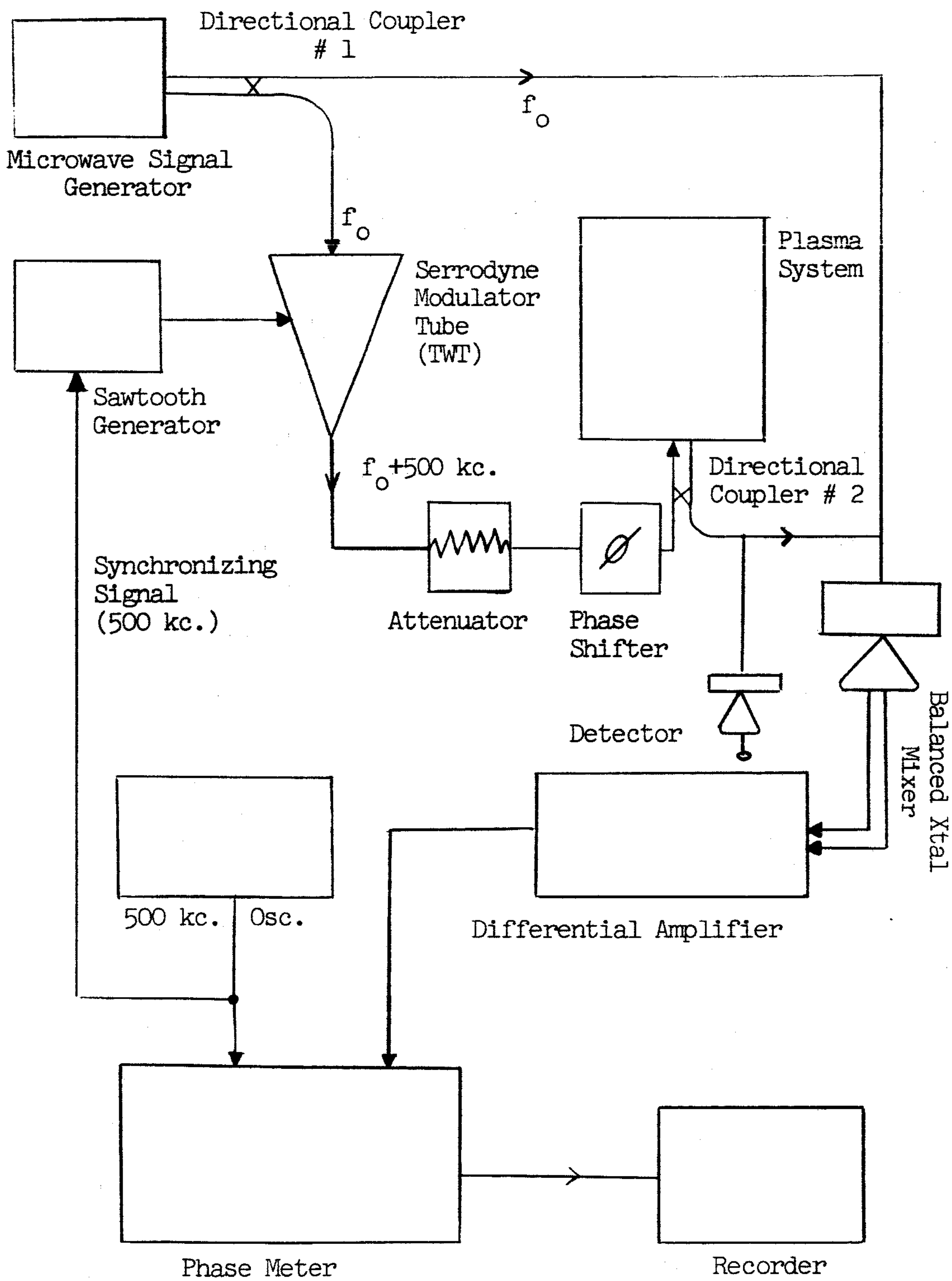


Fig. 8

When the relative phase is slowly varying in time, a phase meter of the type offered commercially by AD-YU is desriable. On the other hand, for rapidly varying phase, the comparison of signals can be made on a dual-beam oscilloscope.

The magnitude of the reflection coefficient is obtained by applying a portion of the reflected signal from directional coupler # 2 to a detector. The presentation of the square law detector signal on an oscilloscope gives the square of the reflection coefficient magnitude as a function of time.

The technique just described here to measure the phase of the reflection coefficient is known as the Serrodyne technique[16] and has been used to study phase shifts in such devices as traveling-wave tubes[17] and ferrites[18] when the phase shift between two signals of widely varying signal strengths was desired.

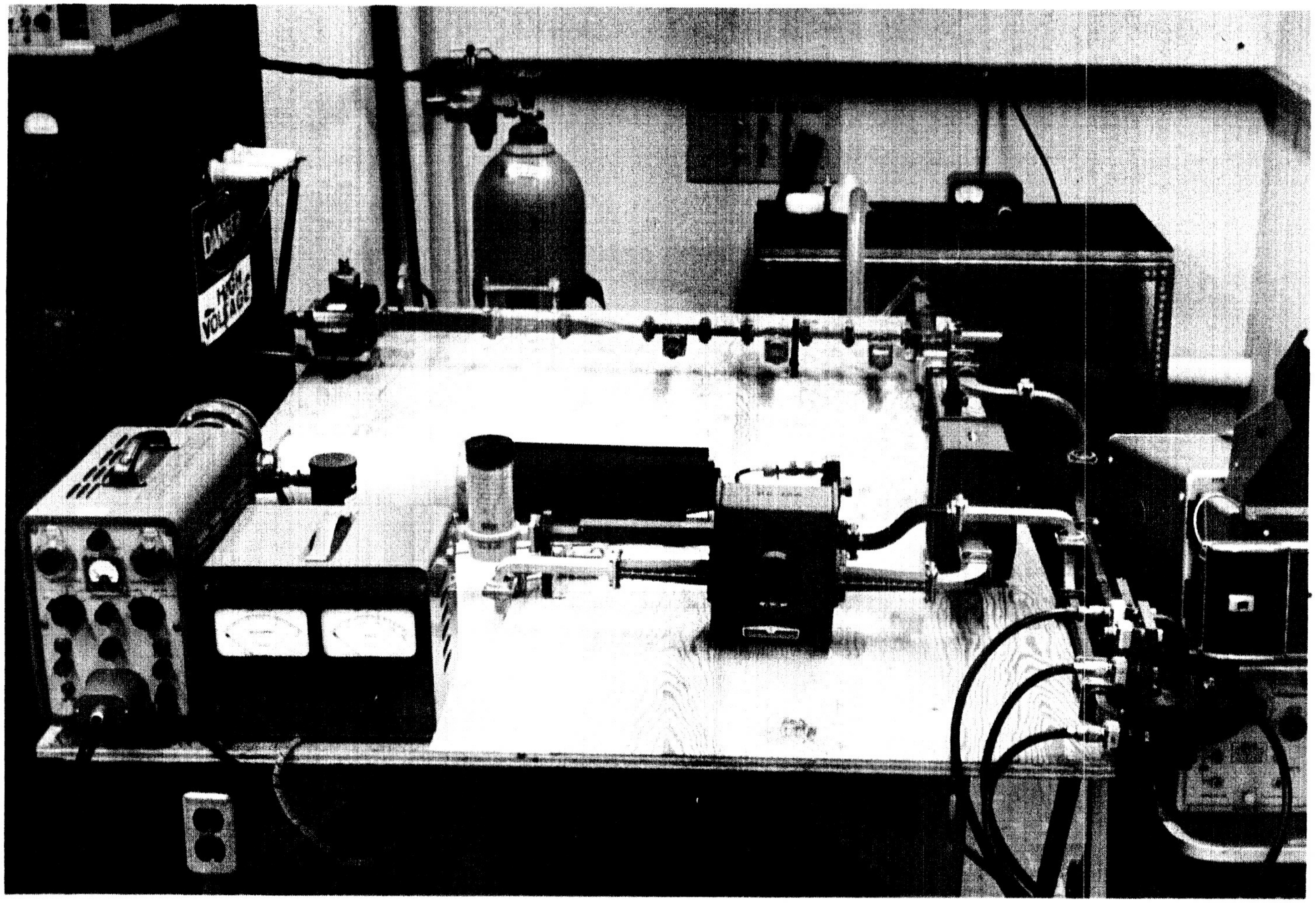


Fig. 9. The experimental apparatus.

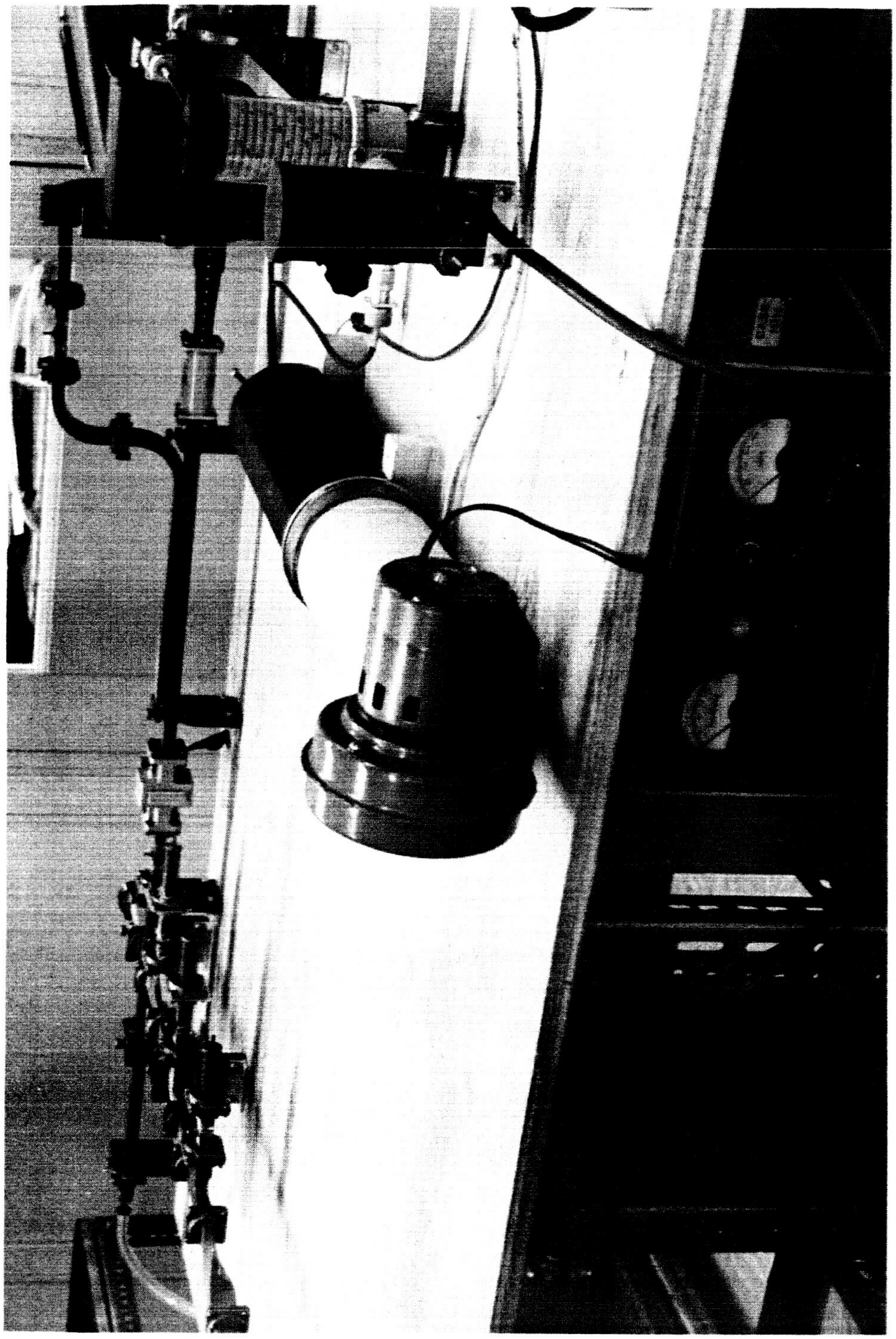


Fig. 10. The plasma ionization system.

CHAPTER IV

EXPERIMENTAL RESULTS

4.1 Experimental Conditions

An experiment was conducted using the apparatus described in Chapter III with the following operating conditions:

Item	Ionizing Source	Probe Source
Source	Magnetron	Klystron
Type	RK-6249 A	725/AB
Frequency	9.375 kmc.	8.254 kmc.
Operation	Pulsed	Continuous
Peak power out	100 kw.	10 mw.
Pulse duration	1μ sec.	———
Repetition rate	125 pps.	———

Plasma container dimensions-I.D.	8in.x.775in.x.275in.
Gas	Argon
Neutral pressure	Variable
Waveguide mode	TE_{10}
Detector output level for complete reflection	.185 volts

4.2 Experimental Data

Initial measurements were made on the magnitude of the reflection coefficient for the ionization and decay of the plasma distribution. The oscilloscope traces shown in Figs. 11, 12, 13, and 14 are the outputs of the square law detector, so the graphs are proportional to the square of the reflection coefficient magnitude. The level of complete reflection for this data is .185 volts.

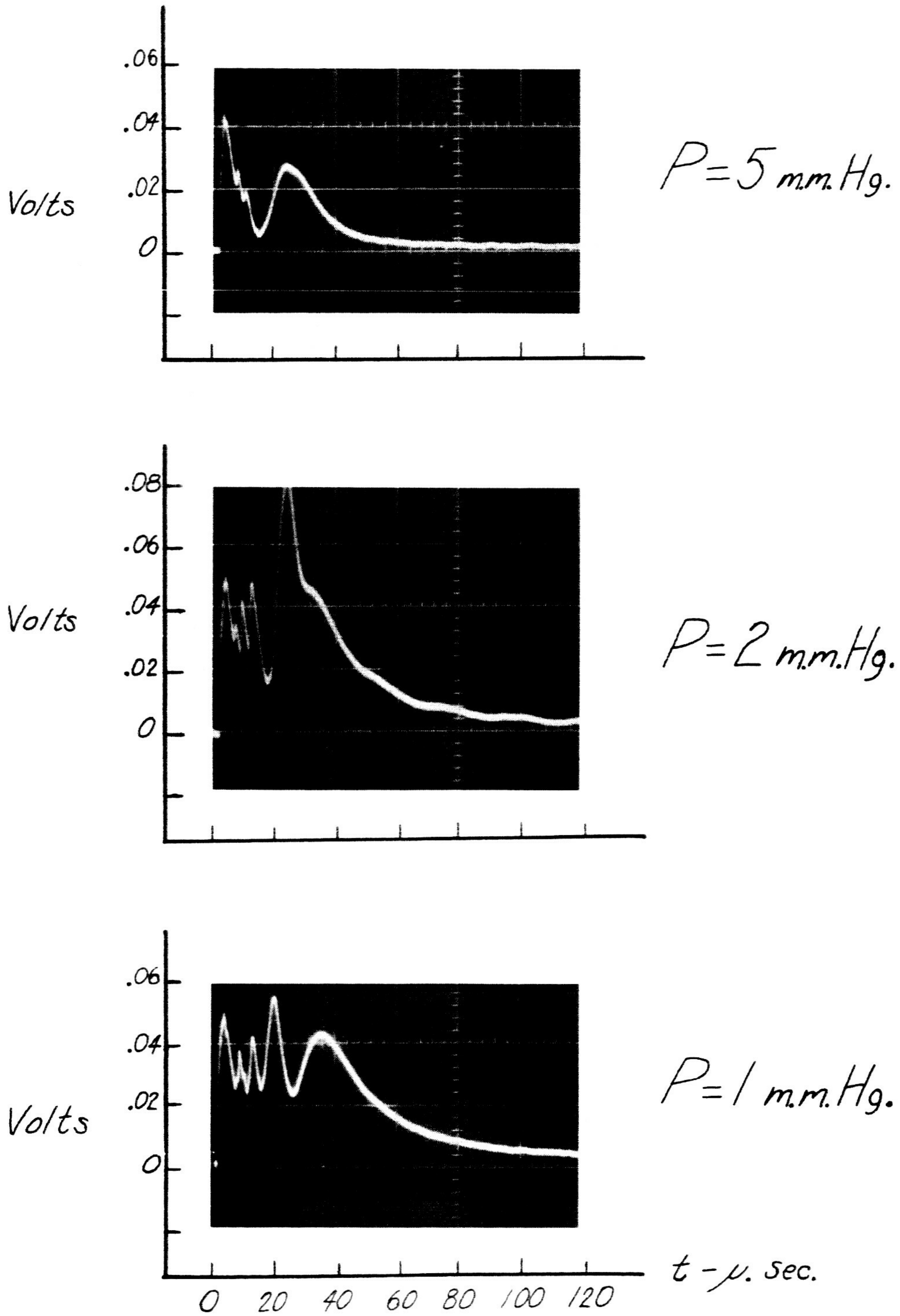


Fig. 11. Detector output for varying neutral gas pressures.

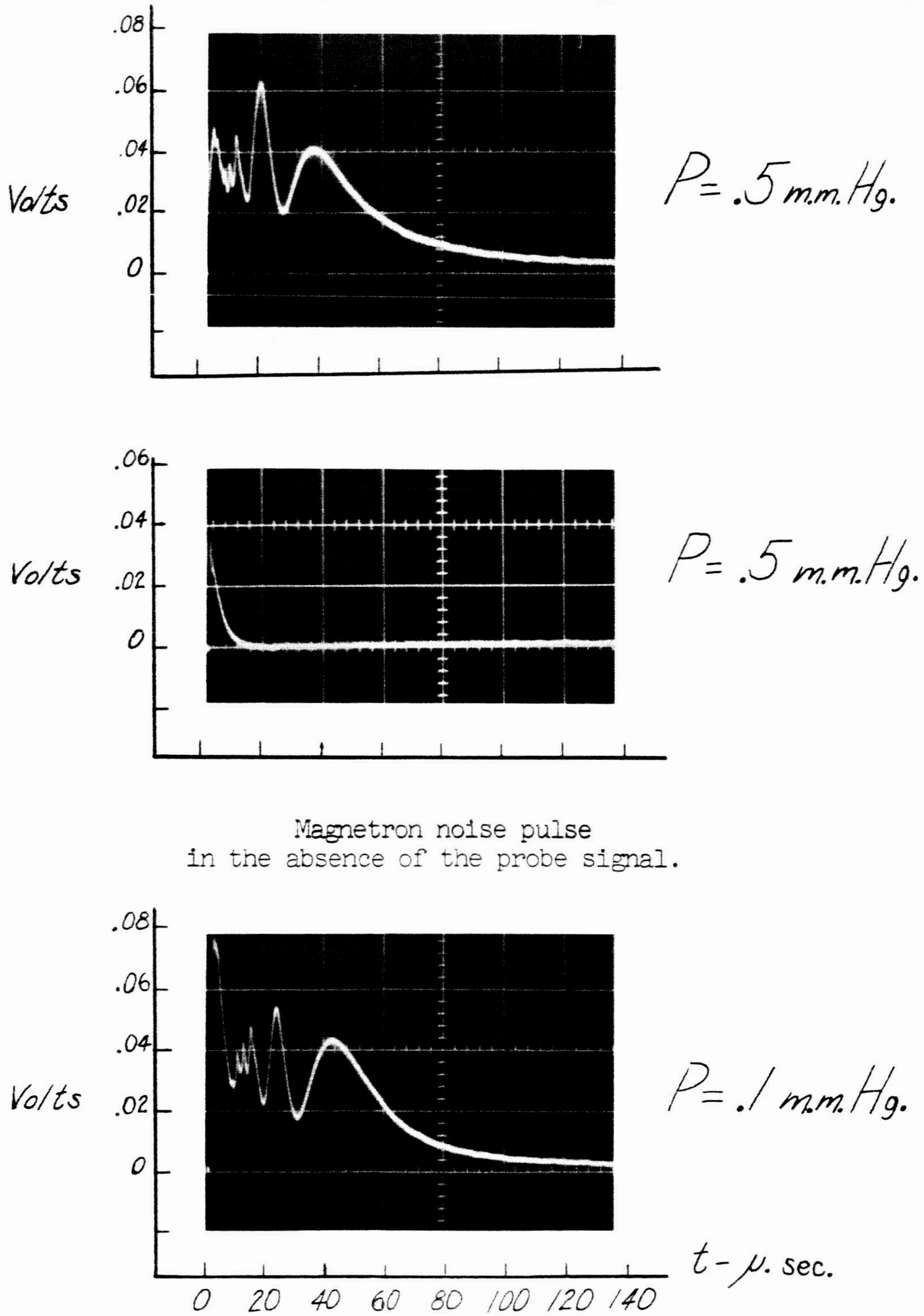


Fig. 12. Detector output for varying neutral gas pressures.

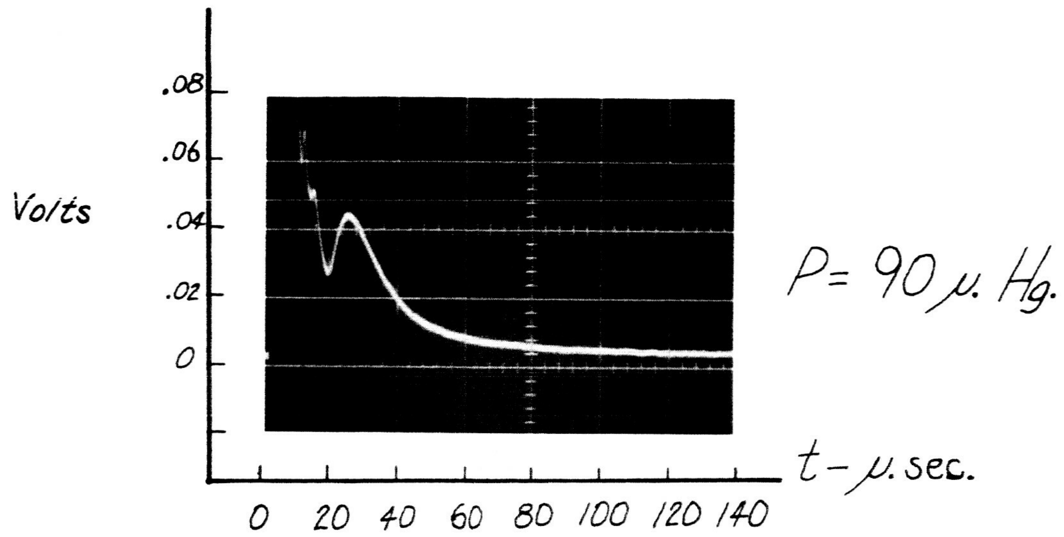


Fig. 13. Detector output versus time.

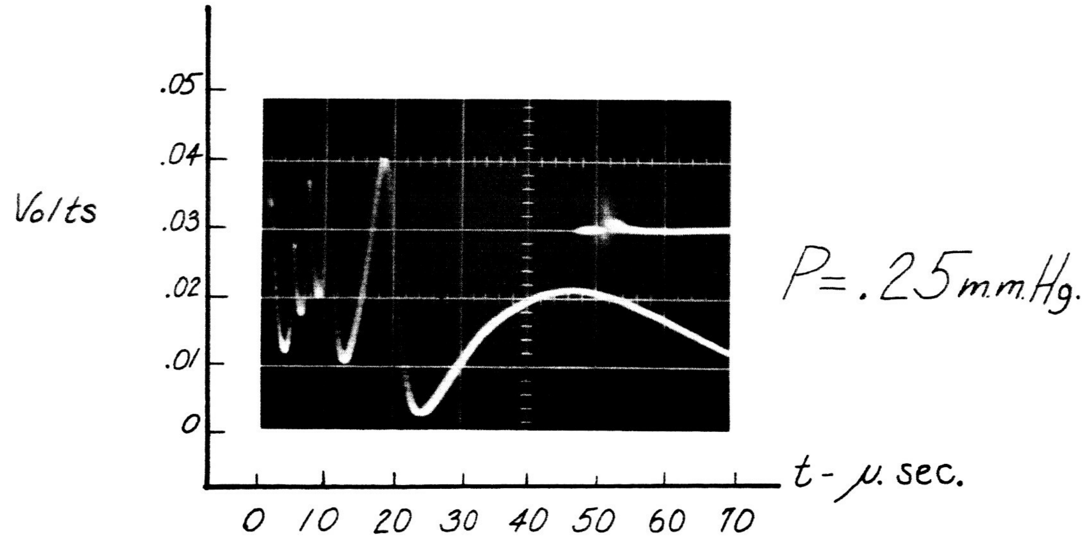


Fig. 14. A typical trace showing detector output for the reflected signal. The magnetron noise pulse in the absence of the signal is shown to the same magnitude scale in the upper right hand corner.

4.3 Discussion and Conclusions

The experimental results of Figs. 11, 12, 13, and 14, show the presence of a number of reflection peaks and minima as the plasma distribution decays in time. These peaks and minima are explained in Appendix B by a simple transmission line theory analogy to waveguides when the plasma distribution is assumed to be spatially uniform throughout the plasma container. This approximation of a decaying uniform plasma is a reasonable one since the plasma cavity is several wavelengths long for the magnetron frequency and the magnetron standing wave pattern tends to ionize the plasma uniformly throughout the plasma container. Also, based on this uniform initial density, the solution for the density obtained from Eq. (2-27) reduces to the first order mode in the transverse directions ($n = m =$ one) due to the fact that, for the observed times of interest ($t > 5$ μ.sec.) and reasonable diffusion coefficient values ($D_{xx} = D_{yy} = D_{zz} \gtrsim 10^3$), the higher order density modes have decayed to insignificant amplitudes. The z-dependent solution remains essentially uniform for the observed times because the exponential loss factor is extremely small for small values of (l). This density mode solution then may be approximated by a decaying uniform distribution for calculating the electromagnetic effects.

On the basis of this simple theory, one would expect the magnitude of the peak reflection to decrease as the plasma decays in time and the minima to remain at the zero level. The fact that this was not experimentally observed is explained by the presence of the effective plasma collision frequency which gives rise to a complex permittivity and a lossy dielectric slab. A detailed discussion of this theoretical problem is given in Appendix B.

The experiment is not very satisfactory for accurate diffusion measurement because of the necessary assumptions made regarding boundary conditions, neglect of recombination, and uniform densities; however, the experiment (for $\omega_p^2 < \omega^2$) provides a good, simple method of obtaining an estimate for the diffusion coefficient. (See details in Appendix B).

It is clear from the results of this experiment that the plasma created was not dense enough to give reflection coefficient magnitudes near unity which indicates that the local plasma frequency ω_p was smaller than the probe signal frequency ω throughout the experiment. Further study of this problem should be made with higher initial densities of plasma, possibly by considering a gas discharge tube as a source for the initial plasma density so that the method of Doppler shift may be evaluated as a useful means of measuring plasma diffusion. Also, studies of diffusion with steady magnetic fields could be made.

Calculations made in Appendix B indicate that a typical value for the diffusion coefficient in Argon at a time of 40 μ.seconds after the ionization is on the order of 1.85×10^3 cm^2 per sec. This value is rather high for ambipolar diffusion in Argon which seems to indicate that for the times of observation the plasma is not in equilibrium and the diffusion loss is a combination of electron and ambipolar diffusion.

APPENDIX A

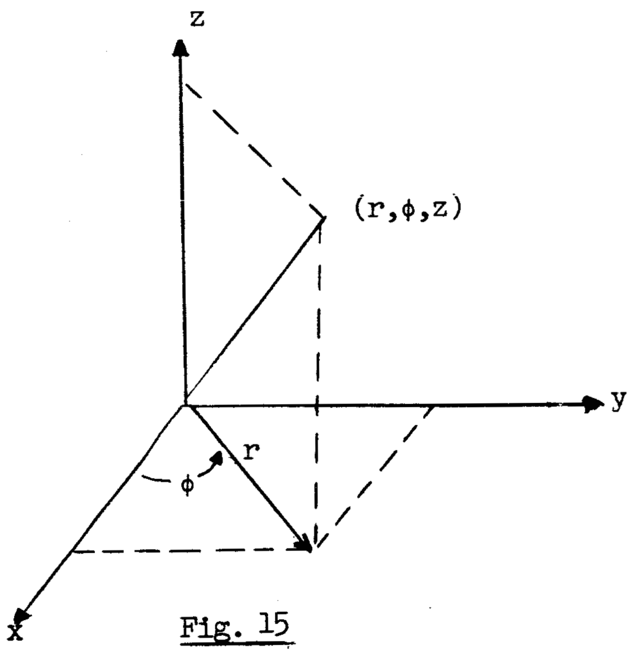


Fig. 15

Diffusion in a Cylindrical Geometry

In cylindrical coordinates, $\nabla.(\overline{D}.\nabla n)$ is written

$$D_{rr}(\partial^2 n/\partial r^2 + 1/r \partial n/\partial r) + D_{\phi\phi}(1/r^2 \partial^2 n/\phi^2)$$

$$+ D_{zz}(\partial^2 n/\partial z^2) + (D_{r\phi} + D_{\phi r})(1/r \partial^2 n/\partial\phi\partial r)$$

$$+ (D_{rz} + D_{zr})(\partial^2 n/\partial z\partial r) + (D_{\phi z} + D_{z\phi})(1/r \partial^2 n/\partial\phi\partial z)$$

$$+ \underline{D_{rz}(1/r \partial n/\partial z)}\ , \qquad \text{(A-1)}$$

where in general

$$\overline{D} = \begin{vmatrix} D_{rr} & D_{r\phi} & D_{rz} \\ D_{\phi r} & D_{\phi\phi} & D_{\phi z} \\ D_{zr} & D_{z\phi} & D_{zz} \end{vmatrix} . \tag{A-2}$$

If a uniform steady magnetic field is directed along the z axis, then it is easily shown by an argument similar to that given for $\overline{D}$ in a rectangular system that

$$D_{r\phi} = -D_{\phi r} \quad ; \quad D_{rz} = D_{zr} = D_{\phi z} = D_{z\phi} = 0 \tag{A-3}$$

and (A-1) reduces to an expression involving D_{rr}, $D_{\phi\phi}$, and D_{zz}, the coordinate diffusion coefficients.

$$\begin{aligned} \nabla.(\overline{D}.\nabla n) &= D_{rr}(\partial^2 n/\partial r^2 + 1/r\partial n/\partial r) \\ &+ D_{\phi\phi}(1/r^2 \partial^2 n/\partial\phi^2) \\ &+ D_{zz}(\partial^2 n/\partial z^2) . \end{aligned} \tag{A-4}$$

Also, for the magnetic field $\underline{B} = B_o \underline{a}_r$,

$$\begin{aligned} &D_{r\phi} = D_{\phi r} = D_{zr} = D_{rz} = 0 ; \\ &D_{\phi z} = -D_{z\phi} \end{aligned} \tag{A-5}$$

and $\nabla.(\overline{D}.\nabla n)$ again reduces to (A-4) . Any other magnetic field configurations such as $\underline{B} = B_o \underline{a}_\phi$ or directions other than $\underline{a}_r$ and $\underline{a}_z$ will produce the term $D_{rz}(1/r\partial n/\partial z)$ and disrupt the symmetry of the equation except in those cases where the plasma density is uniform along the z axis. For the case of uniformity along z, $\partial n/\partial z = 0$, and (A-1) always reduces to

$$\nabla.(\bar{D}.\nabla n) = D_{rr}(\partial^2 n/\partial r^2 + 1/r\partial n/\partial r)$$
$$+ D_{\phi\phi}(1/r^2 \partial^2 n/\partial\phi^2) \ . \qquad \text{(A-6)}$$

for any uniform magnetic field configuration, $\underline{B} = B\,\underline{a}_\phi$ or $B\,\underline{a}_r$ or $B\,\underline{a}_z$ or any linear combination of these. When a particular symmetry of the distribution function exists or a field orientation exists that makes the cross term $D_{rz}(1/r\partial n/\partial z)$ vanish, the solution of the diffusion problem is possible by separation of variables.

Solutions of Diffusion in a Cylindrical Geometry

It is assumed in all of the following solutions that even symmetry exists over the ϕ coordinate so that any expasnion of the solution involves $\cos n\phi$. For the general case where no symmetry exists over the ϕ coordinates, the $\cos n\phi$ terms must be replaced by $(A_n \cos n\phi + B_n \sin n\phi)$.

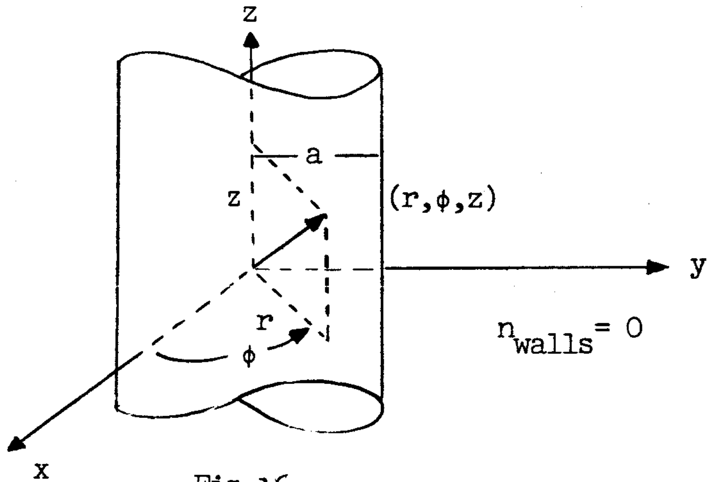


Fig. 16

Infinite Cylinder. Consider the geometry of Fig. 7 for an infinitely long cylinder of radius a where (2-1) is to be solved in cylindrical coordinates. It is also assumed that the magnetic field configuration is such that (2-1) can be written

$$D_{rr}(\partial^2n/\partial r^2 + 1/r\,\partial n/\partial r) + D_{\phi\phi}(1/r^2\,\partial^2n/\partial\phi^2)$$
$$+ D_{zz}(\partial^2n/\partial z^2) + \nu n = \partial n/\partial t \qquad \text{(A-7)}$$

where ν, again, is the difference between the ionization collision frequency and the attachment collision frequency. The solution of this problem will proceed along the same lines as for the rectangular case.

The Laplace transform of (A-7) gives

$$D_{rr}(\partial^2N/\partial r^2 + 1/r\partial N/\partial r) + D_{\phi\phi}(1/r^2\partial^2N/\partial\phi^2)$$

$$+ D_{zz}(\partial^2N/\partial z^2) + (\nu - s)\,N = -N_o \qquad \text{(A-8)}$$

where

$$N = \int_0^\infty n(r,\phi,z,t)\exp[-st]dt\ ,$$

$$N_o = n(r,\phi,z,0)\ .$$

If $D_{rr} = D_{\phi\phi}$, n could be expanded in a series of modes for the problem, $\cos n\phi\, J_n(p_{nm}\, r)\ \ h(z,t)$. However, the solutions for $D_{rr} \neq D_{\phi\phi}$ can be expressed in series form if the Bessel function solutions of the first kind are modified. The solution can be written*

$$n(r,\phi,z,t) = \sum_{\substack{n=0\\m=0}}^{\infty} \cos n\phi J_{(kn)}[p_{(kn)m}r]\ h(z,t)$$

$$\text{(A-9)}$$

$$N_o = \sum_{\substack{n=0\\m=0}}^{\infty} b_{nm} \cos n\phi J_{(kn)}[p_{(kn)m}r]\ q(z)$$

where $k = +(D_{\phi\phi}/D_{rr})^{1/2}$, $p_{(kn)m}$ is defined by $J_{(kn)}[p_{(kn)m}a]=0$.

---

*No attempt will be made to prove completeness of this set of functions since most diffusion problems of interest require only a finite number of expansion terms or $k=1$.

If the assumptions of (A-9) are applied to (A-8) and terms of the series are equated, the following relation is obtained:

$$-D_{rr}[p_{(kn)m}{}^2 - k^2n^2/r^2]\,\mathcal{h}(z,s) - D_{\phi\phi}[n^2/r^2]\mathcal{h}(z,s)$$

$$+D_{zz}[\partial^2\mathcal{h}(z,s)/\partial z^2] + [\nu - s]\mathcal{h}(z,s) = -c_{nm}\,q(z) \qquad \text{(A-10)}$$

where $c_{nm} = b_{nm}/a_{nm}$, $\mathcal{h}(z,s)$ is the Laplace transform of $h(z,t)$.

Taking the Fourier transform over the z variable gives

$$[D_{rr}\,p_{(kn)m}{}^2 + D_{zz}\omega^2 - (\nu - s)]\,\mathcal{H}(\omega, s) = \phi_{nm}(\omega) \qquad \text{(A-11)}$$

where $\phi_{nm}(\omega)$ is the Fourier transform of $c_{nm}\,q(z)$, $\mathcal{H}(\omega, s)$ is the Fourier transform of $\mathcal{h}(z,s)$.

The inverse Laplace and Fourier transforms of (A-11) yields

$$H(\omega,t) = \exp[(\nu - D_{rr}p_{(kn)m}{}^2)t\,]\,\exp[-D_{zz}\omega^2 t]\phi_{nm}(\omega) \qquad \text{(A-12)}$$

$$h(z,t) = \exp[(\nu - D_{rr}p_{(kn)m}{}^2)t\,]\int_{-\infty}^{\infty}\exp[j\omega z - D_{zz}\omega^2 t]\phi_{nm}(\omega)\,d\omega\ . \qquad \text{(A-13)}$$

As in the rectangular case, the Green's function solution is desired so that it is assumed

$$N_o = \frac{\delta(r-r')}{r}\,\delta(z-z')\,\delta(\phi-\phi') \qquad \text{(A-14)}$$

So that

$$q(z) = \delta(z-z') \,.$$

If (A-9) and (A-14) are equated,

$$b_{nm'} = 2\,J_{(kn)}[p_{(kn)m}\, r']\cos n\phi'/a^2\pi[J'_{(kn)}\{p_{(kn)m}\, a\}]^2, \; n\neq 0$$

$$= \quad J_o(p_{om} r')/a^2\pi[J_1(p_{om}\, a)]^2, \quad n=0 \qquad \text{(A-15)}$$

where $J'_{(kn)}[\ \ ]$ denotes differentiation of $J_{(kn)}[\ \ ]$ with respect to $p_{(kn)m}\, r$. Equation (A-13) can be integrated to give

$$h(z,t) = \exp[(\nu - D_{rr}\, p_{(kn)m}{}^2)t][c_{nm}/2(\pi D_{zz} t)^{1/2}]$$

$$\exp\{-[(z-z')/2(D_{zz}t)^{1/2}]^2\}. \qquad \text{(A-16)}$$

The complete Green's function solution of (A-7) then is

$$G(r|r',\phi|\phi',z|z',t) =$$

$$\sum_{\substack{n=0\\ m=0}}^{\infty} b_{nm'} \cos n\phi\; J_{(kn)}[p_{(kn)m}\, r]\exp[(\nu - D_{rr}\, p_{(kn)m}{}^2)t]$$

$$[1/2(\pi D_{zz} t)^{1/2}]\exp\{-[(z-z')/2(D_{zz}t)^{1/2}]^2\}. \qquad \text{(A-17)}$$

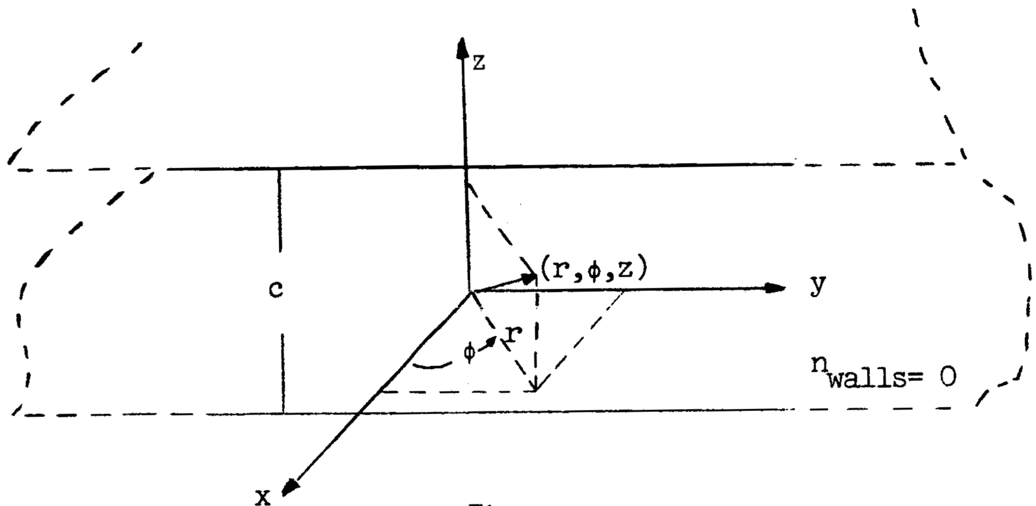


Fig. 17

Cylinder of Infinite Radius and Finite Length. Next, the problem of the infinite radius cylinder as in Fig. 8 will be considered. The solution to this problem has already be given on page 13 of Chapter 2,(2-35), in rectangular coordinates.

The solution to (A-7) is desired subject to the same boundary conditions, namely $n_{walls} = 0$. The Laplace transform in the time domain gives (A-8):

$$D_{rr}(\partial^2N/\partial r^2 + 1/r\partial N/\partial r) + D_{\phi\phi}(1/r^2\partial^2N/\partial\phi^2)$$
$$+ D_{zz}(\partial^2N/\partial z^2 + (\nu - s)N = -N_o$$

where N is again the Laplace transform of $n(r,\phi,z,t)$, N_o is $n(r,\phi,z,0)$. It is also assumed that N and N_o can be expanded in series of the form

$$n(r,\phi,z,t) = \sum_{\substack{n=0\\m=0}}^{\infty} a_{nm} \cos n\phi \sin(m\pi z/c)\, R(r,t) \;, \qquad \text{(A-18)}$$

$$n(r,\phi,z,0) = N_o = \sum_{\substack{n=0\\m=0}}^{\infty} b_{nm} \cos n\phi \sin(m\pi z/c)\, q(r) \qquad \text{(A-19)}$$

where $q(r)$ is the initial radial dependence of N_o .

Applying these assumptions to (A-8) and equating terms of the series gives

$$\begin{aligned} &D_{rr}(\partial^2\underline{R}/\partial r^2 + 1/r\partial\underline{R}/\partial r) + [D_{\phi\phi}(-n^2/r^2) \\ &- D_{zz}(m\pi/c)^2 + (\nu - s)]\,\underline{R} = -c_{nm}\, q(r) \end{aligned} \qquad \text{(A-20)}$$

where $\underline{R}$ is the Laplace transform of $R(r,t)$, $c_{nm} = b_{nm}/a_{nm}$.

In order to suppress the r variable, the Hankel transform pair is used

$$\begin{aligned} P(\omega) &= \int_0^\infty \psi(r)\, J_{(kn)}(\omega r) r\, dr \\ \psi(r) &= \int_0^\infty P(\omega)\, J_{(kn)}(\omega r) \omega\, d\omega \;. \end{aligned} \qquad \text{(A-21)}$$

The Hankel transform of (A-20) reduces in its final form to

$$[(s - \nu) + D_{zz}(m\pi/c)^2 + D_{rr}\omega^2]\mathcal{R} = K(\omega) \qquad \text{(A-22)}$$

where $\mathcal{R}$ is the Hankel transform of $\underline{R}$ and $K(\omega)$ is the Hankel transform of $c_{nm}\, q(r)$.

The expression for $\mathcal{R}$ then is

$$\mathcal{R} = K(\omega)/(s - \nu) + D_{zz}(m\pi/c)^2 + D_{rr}\omega^2 \tag{A-23}$$

The inverse Laplace and Hankel transforms of (A-23) yield

$$R(r,t) = \exp[(\nu - D_{zz}(m\pi/c)^2)]t$$

$$\int_0^\infty K(\omega)\, \exp[-D_{rr}\omega^2 t]\, J_{(kn)}(\omega r)\omega d\omega \quad . \tag{A-24}$$

In order to obtain the Green's function solution for the problem, it is assumed that the initial distribution N_o is a delta function

$$N_o = n(r,\phi,z,0) = \delta\frac{(r-r')}{r}\,\delta(\phi-\phi')\delta(z-z') \tag{A-25}$$

For this initial distribution, $K(\omega)$ then, is

$$K(\omega) = \int_0^\infty \delta\frac{(r-r')}{r}\, r\, J_{(kn)}(\omega r)\, dr$$

$$= J_{(kn)}(\omega r') \tag{A-26}$$

and

$$R(r,t) = \exp[(\nu - D_{zz}(m\pi/c)^2)t]$$

$$\int_0^\infty J_{(kn)}(\omega r')\, J_{(kn)}(\omega r)\, \exp[-D_{rr}\omega^2 t]\omega d\omega \quad . \tag{A-27}$$

The integral solution has been tabulated in "Tables of Integral Transforms", Bateman Manuscript Project, McGraw-Hill, Volume 2, p. 51, 1954. It is given as:

$$1/(2D_{rr}t)\ \exp[-(r'^2 + r^2)/(4D_{rr}t)]\ I_{(kn)}(rr'/2D_{rr}t) \qquad \text{(A-28)}$$

where $I_\nu(\eta)$ is the modified Bessel function of the first kind of order ν. $R(r,t)$ is written

$$R(r,t) = \exp[(\nu - D_{zz}(m\pi/c)^2)t]\,1/(2D_{rr}t)\ \exp[-(r'^2 + r^2)/(4D_{rr}t)]$$
$$I_{(kn)}(rr'/2D_{rr}t)\ . \qquad \text{(A-29)}$$

The final Green's function solution, then, is

$$n(r,\phi,z,t) = \sum_{\substack{n=0\\m=0}} b_{nm'}\cos n\phi\ \sin(m\pi z/c)\ \exp[(\nu - D_{zz}(m\pi/c^2)t]$$
$$1/(2D_{rr}t)\ \exp[-(r'^2 + r^2)/(4D_{rr}t)]I_{(kn)}(rr'/2D_{rr}t)$$
$$\text{(A-30)}$$

where $b_{nm'} = (2/\pi c)\cos n\phi' \sin(m\pi z'/c)$ for $n \neq 0$,

$= (1/\pi c)\sin(m\pi z/c)$ for $n = 0$. (A-31)

Green's Function Solution for Unbounded Cylindrical Region.

The unbounded solution in cylindrical coordinates can easily be obtained by using the results of (A-30) and (2-30). The resultant solution is $G(r|r', \phi|\phi', z|z', t$

$$= \sum_{n=0}^{\infty} b_{n'} \cos n\phi \; 1/(2D_{rr}t) \exp[-(r'^2 + r^2)/(4D_{rr}t)] \exp[\nu t]$$

$$I_{(kn)}(rr'/2D_{rr}t) \; 1/2(\pi D_{zz}t)^{1/2} \exp[-[(z-z')/2(D_{zz}t)^{1/2}]]^2$$

$$t > 0 \qquad \text{(A-32)}$$

$$= 0 \,, \quad t < 0$$

where

$$b_{n'} = 1/\pi \cos n\phi' \quad , \quad n \neq 0$$

$$= 1/2\pi \,, \qquad n = 0 \,. \qquad \text{(A-33)}$$

<u>Green's Function Solution for a Bounded Cylindrical Region.</u>

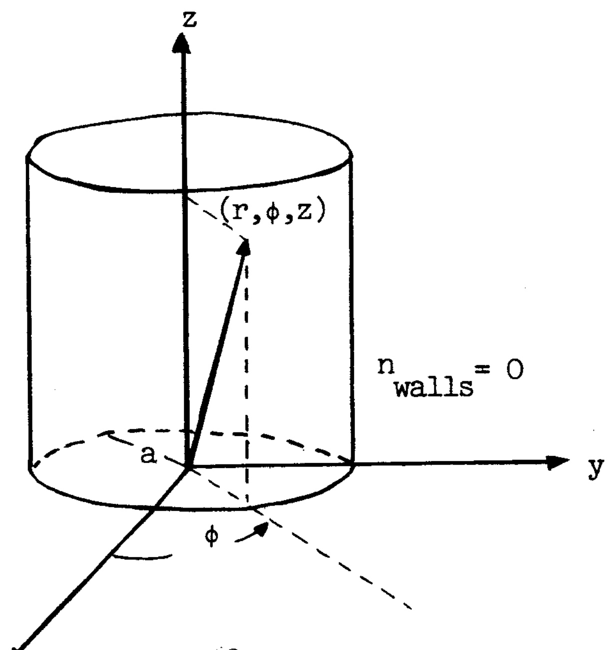


Fig. 18

The Green's function solution for the geometry of Fig. 9 can be expanded in a series of modes for the cylindrical geometry:

$$G(r|r', \phi|\phi', z|z', t) =$$

$$\sum_{\substack{n=0\\m=0}}^{\infty} b_{nm'} \cos n\phi \; J_{kn}(p_{kn\,m}\, r) \sin(m\pi z/c)$$

$$\exp[(\nu - D_{zz}(m\pi/c)^2 - D_{rr}(p_{kn\,m})^2)t] \quad , \quad t > 0$$

$$= 0 \quad , \quad t < 0 \qquad \text{(A-34)}$$

where $b_{nm'}$ is given as

$$b_{nm'} = \frac{4}{\pi a^2 c[J_{kn'}(p_{kn\,m}\, a)]^2} \cos n\phi' \sin(m\pi z'/c)$$

$$J_{kn}(p_{kn\,m}\, r') \; , \quad n \neq 0 \; ,$$

$$= \frac{2}{\pi a^2 c[J_1(p_{om}\, a)]^2} \sin(m\pi z'/c) \; J_o(p_{om} r')$$

$$n = 0 \; . \qquad \text{(A-35)}$$

Diffusion in a Spherical Geometry

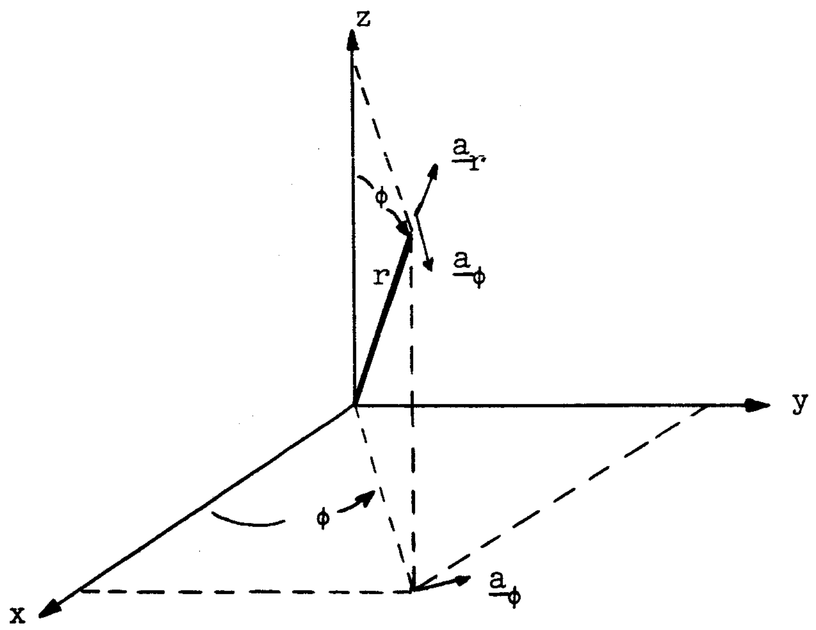


Fig. 19

In spherical coordinates, $\nabla.(\overline{D}.\nabla n)$ is expanded as

$$D_{rr}(\partial^2 n/\partial r^2 + 2/r\partial n/\partial r) + (D_{r\phi} + D_{\phi r})(1/r \sin\theta\partial^2 n/\partial\phi\partial r)$$

$$+ \underline{D_{r\phi}(1/r^2 \sin\theta\partial n/\partial\phi\ )} + D_{\phi\phi}(\cos\theta/r^2 \sin\theta\partial n/\partial\theta + 1/r^2\partial^2 n/\partial\theta^2)$$

$$+ (D_{\theta\phi} + D_{\phi\theta})(1/r^2 \sin\theta\partial^2 n/\partial\theta\partial\phi) + D_{\phi\phi}(1/r^2 \sin^2\theta\partial^2 n/\partial\phi^2)\ . \tag{A-36}$$

If, for the spherical case, $\underline{B} = B_o\, \underline{a}_\phi$ or $B_o\, \underline{a}_r$, (A-36) reduces to

$$\nabla.(\bar{D}.\nabla n) = D_{rr}(\partial^2 n/r + 2/r\partial n/\partial r)$$
$$+ D_{\theta\theta}(\cos\theta/r^2\sin\theta\partial n/\partial\theta + 1/r^2\partial^2 n/\partial\theta^2)$$
$$+ D_{\phi\phi}(1/r^2\sin\theta\partial^2 n/\partial\phi^2) \ . \qquad \text{(A-37)}$$

Also, if symmetry exists about the z axis, $\partial n/\partial\phi = 0$ and equation (A-36) reduces to

$$\nabla.(\bar{D}.\nabla n) = D_{rr}(\partial^2 n/\partial r^2 + 2/r\partial n/\partial r)$$
$$+ D_{\theta\theta}(\cos\theta/r^2\sin\theta\partial n/\partial\theta + 1/r^2\partial^2 n/\partial\theta^2) \qquad \text{(A-38)}$$

for any linear combination of uniform magnetic field components $\underline{B} = B_1\,\underline{a}_r + B_2\,\underline{a}_\theta + B_3\,\underline{a}_\phi + B_4\,\underline{a}_z$.

The generalized solutions are difficult to obtain in terms of known functions, and no attempt at the solutions will be made here.

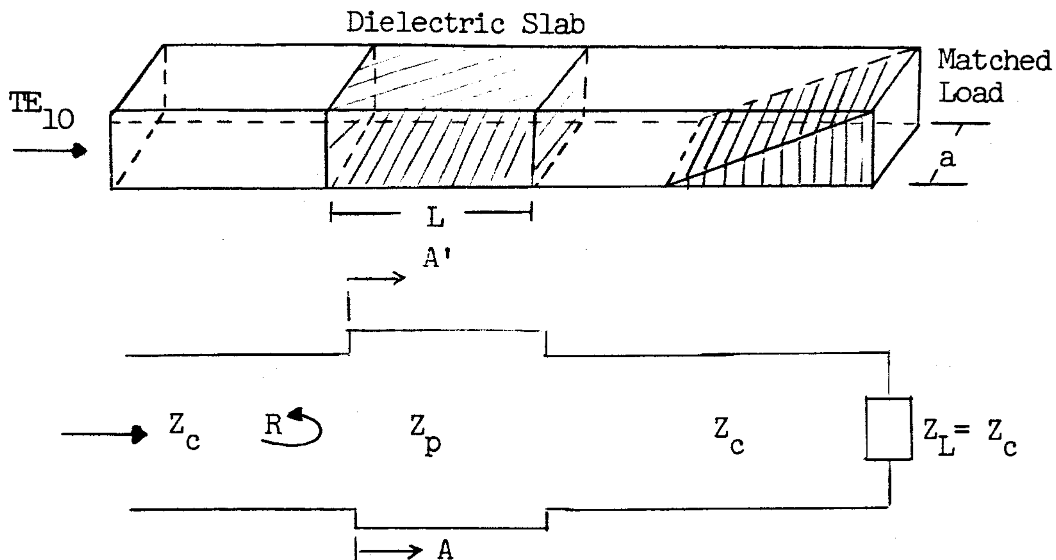
APPENDIX B

REFLECTION FROM A PLASMA SLAB IN A WAVEGUIDE

Waveguide-Transmission Line Analogy

The problem of transmission and reflection of a dominant mode in a waveguide in which the propagation constant is only a function of the longitudinal direction can be treated by solving the analogous problem by transmission line theory since the equation governing transmission along the length of the two systems is the same. The wave impedance for the mode in the waveguide is directly analogous to the characteristic impedance for a transmission line.

Consider the following transmission line analogy to the problem of a waveguide which is perfectly matched but contains a uniform slab of dielectric of length L .



Z_c = Characteristic impedance or wave impedance for the waveguide in free space.

Z_p = Characteristic impedance or wave impedance for the dielectric section.

Fig. 20. Transmission Line Analogy

According to transmission line theory, the impedance seen looking toward the load at section A-A' is

$$Z = \frac{Z_p(Z_L + Z_p \tanh \gamma L)}{(Z_p + Z_L \tanh \gamma L)} \tag{B-1}$$

$$= \frac{Z_p(1 + \overline{Z}_p \tanh \gamma L)}{(\overline{Z}_p + \tanh \gamma L)}$$

where $\gamma = \alpha + j\beta$ is the complex propagation factor in the dielectric; α is the attenuation constant and β is the propagation constant. $\overline{Z}_p = Z_p/Z_c$ is defined as the characteristic normalized impedance of the slab section.

The corresponding reflection coefficient for the input to the slab section is

$$R = \frac{Z - Z_c}{Z + Z_c} = \frac{(\overline{Z}_p^2 - 1)\tanh \gamma L}{(\overline{Z}_p^2 + 1)\tanh \gamma L + 2\,\overline{Z}_p}$$

$$= \frac{(\overline{Z}_p^2 - 1)}{(\overline{Z}_p^2 + 1) + 2\,\overline{Z}_p \coth \gamma L} \tag{B-2}$$

The propagation factor γ is defined by

$$\gamma = (k_c^2 - \kappa k_o^2)^{1/2} = \alpha + j\beta \tag{B-3}$$

where k_c = (π/a) for the TE_{10} rectangular waveguide mode,

κ = dielectric constant = $\varepsilon/\varepsilon_o$,

k_o = $\omega/c = 2\pi/\lambda_o$,

λ_o = free space wavelength of the probe signal,

ω = radian frequency of the probe signal,

c = speed of light in free space.

The characterisitc impedances are defined by [19]

$$Z_c = (jk_o/\gamma_o)\,Z_o$$

where $Z_o = (\mu_o/\varepsilon_o)^{1/2}$ = characteristic impedance of free space,

$$\gamma_o = (k_c^2 - k_o^2)^{1/2}$$

$$Z_p = (jk_o/\gamma)Z_o \ , \tag{B-4}$$

$$\overline{Z}_p = \gamma_o/\gamma = \frac{(k_c^2 - k_o^2)^{1/2}}{(k_c^2 - \kappa k_o^2)^{1/2}} = a + jb \ .$$

Collisionless Plasma Slab

The problem of finding the maximum and minimum magnitudes of the reflection coefficient for a slab with a complex dielectric constant κ is difficult because of the awkward expressions involved. For a lossless plasma, however, characterized by the dielectric constant $\kappa = (1 - \omega_p^2/\omega^2)$, the problem is simplified. If $(\pi c/a)^2 + \omega_p^2 < \omega^2$ where ω_p is the plasma frequency, and γ_o and $\gamma = j\beta$ are pure imaginary numbers corresponding to a propagating mode in the waveguide, than $\overline{Z}_p$ is a positive real quantity. For this case, $\coth\gamma L = j \cot\beta L$ and

$$R = \frac{(\overline{Z}_p^2 - 1)}{(\overline{Z}_p^2 + 1) + 2j\overline{Z}_p \cot\beta L} \tag{B-5}$$

The relative minimum values for $|R|$ occur for

$$\beta L = m\pi \ ,$$
$$m = 1, 2, 3, \ldots \tag{B-6}$$

and the corresponding value of $|R|_{min}$ for these values of βL is

$$|R|_{min} = 0 . \qquad \text{(B-7)}$$

The relative maxima occur for

$$\beta L = (n + 1/2)\pi ,$$
$$n = 0, 1, 2, 3, \ldots.. \qquad \text{(B-8)}$$

and the corresponding value of the maximum is

$$|R|_{max} = \frac{\overline{Z}_p^2 - 1}{\overline{Z}_p^2 + 1} . \qquad \text{(B-9)}$$

Also, a maximum reflection of unity occurs for the lossless plasma case when $\beta = 0$. This is the cutoff condition for propagation of the waveguide mode in the plasma slab section.

The preceding relations for β give the "resonant" conditions for the plasma slab when the dielectric constant is pure real and the mode is propagating in the plasma section. For a dielectric constant, $\kappa = (1 - \omega_p^2/\omega^2)$, the corresponding relations between ω and ω_p are

$$\underline{|R|_{max}} \qquad \omega^2=\omega_p^2+ \frac{(n+1/2)^2\pi^2c^2}{L^2} + \frac{\pi^2c^2}{a^2} \qquad \text{(B-10)}$$

$$n = 0, 1, 2, 3, \ldots\ldots.$$

$$\omega^2=\omega_p^2 + \frac{\pi^2c^2}{a^2} \quad \text{-- cutoff condition}$$

$$\underline{|R|_{min}} \qquad \omega^2 = \omega_p^2 + \frac{m^2\pi^2c^2}{L^2} + \frac{\pi^2c^2}{a^2} \qquad \text{(B-11)}$$

$$m = 1, 2, 3, \ldots\ldots\ldots \quad .$$

A plot of $|R|$ versus βL is given in Fig. 21.

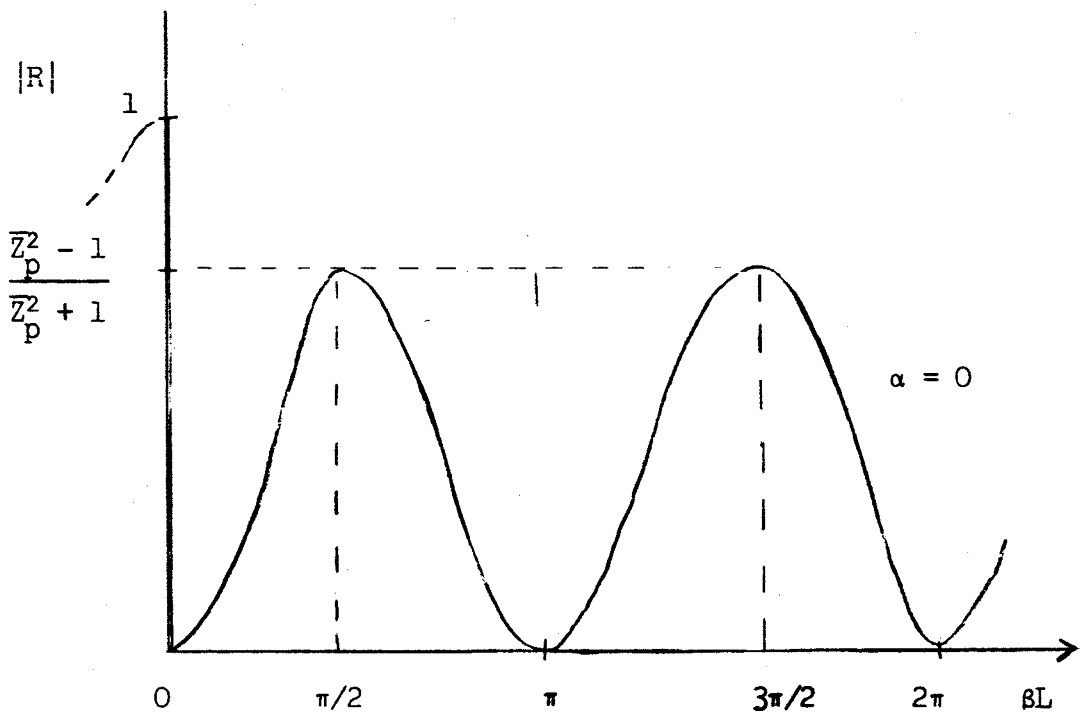


Fig. 21. $|R|$ Versus βL for Lossless Slab.

Plasma Slab with Collisions

For a temperate plasma in which the collisions between the constituent particles cannot be neglected as a loss term, the dielectric constant is usually characterized by

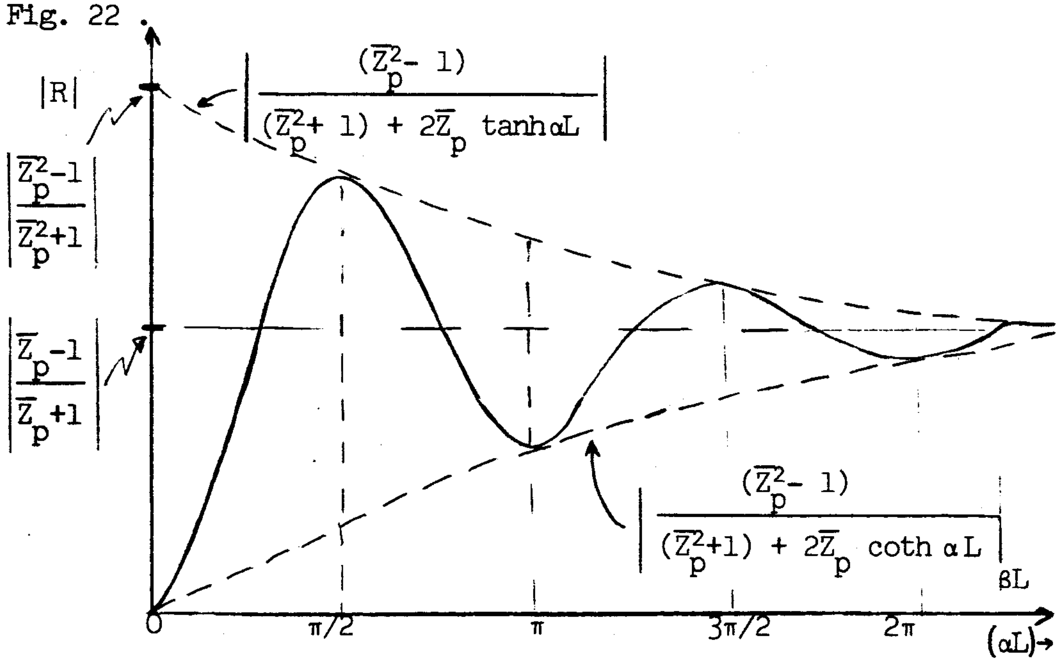
$$\kappa = 1 - \frac{\omega_p^2(1 + j\nu/\omega)}{\omega^2 + \nu^2} \quad , \qquad \text{(B-12)}$$

where ν is the effective collision frequency of the plasma. When the collision frequency ν is small compared to the probe signal frequency ω, the values of βL for maximum or minimum reflection magnitudes do not change appreciably from the collisionless case but the maxima and minima of the reflection coefficient magnitude are reduced to the approximate expressions

$$\underline{|R|_{max}} = \frac{(\overline{Z}_p^2 - 1)}{(\overline{Z}_p^2 + 1) + 2\overline{Z}_p \tanh\alpha L}, \tag{B-13}$$

$$\underline{|R|_{min}} = \frac{(\overline{Z}_p^2 - 1)}{(\overline{Z}_p^2 + 1) + 2\overline{Z}_p \coth\alpha L}. \tag{B-14}$$

A plot of the reflection coefficient magnitude is given in Fig. 22.



<u>Fig. 22.</u> $|R|$ vs. βL for Lossy Slab.

It has been shown that the effects of losses in the dielectric slab or collisions for a plasma slab has the effect of lowering the reflection magnitude peaks and raising of the reflection minima from the zero level. When the effective collision frequency is high, both the levels of the maximum and minimum reflections and the corresponding values of βL are changed appreciably.

Calculations of Resonance Densities and the Diffusion Coefficient

The electron densities at the peaks and nulls shown in Figs. 11 - 14 can be calculated from Eqs. (B-10) and (B-11).

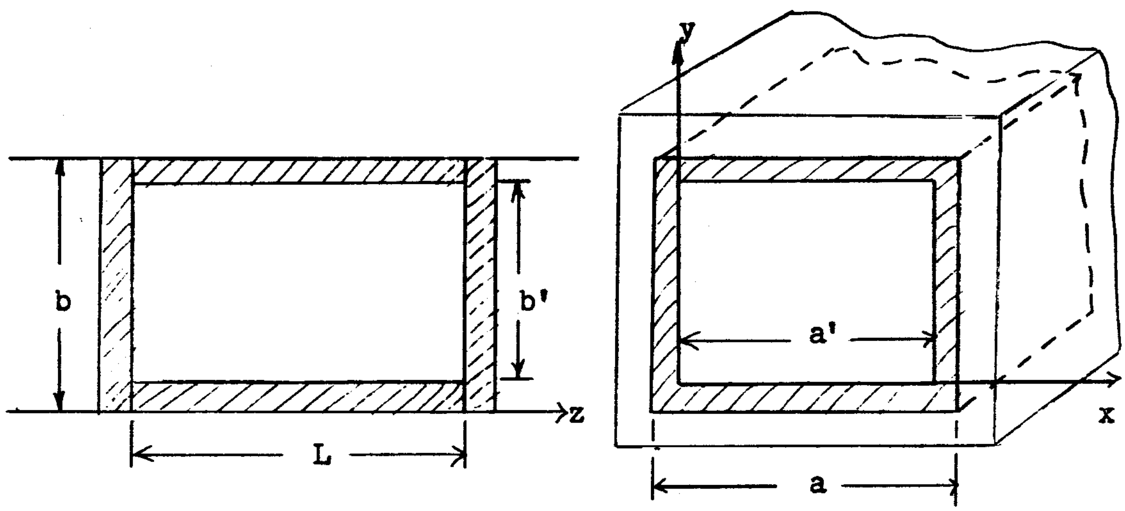


Fig. 23 Waveguide and Plasma Cavity Dimensions.

Waveguide Dimensions - I.D.

$a = .9$ in. $= 2.29$ cm.

$b = .4$ in. $= 1.02$ cm.

Quartz Container Dimensions - I.D.

$a' = .775$ in. $= 1.97$ cm.

$b' = .275$ in. $= .7$ cm

$L = 8$ in. $= 20.32$ cm.

$f = 8.254$ kmc = probe signal frequency

$\omega = 5.19 \times 10^{10}$ radians per second

Peaks [Eq. (B-10)]

$$\omega^2 = \omega_p^2 + (n + \tfrac{1}{2})^2 \frac{\pi^2 c^2}{L^2} + \frac{\pi^2 c^2}{a^2}$$

$$n = 0, 1, 2, 3 \ldots..$$

Minima [Eq. (B-11)]

$$\omega^2 = \omega_p^2 + \frac{m^2 \pi^2 c^2}{L^2} + \frac{\pi^2 c^2}{a^2}$$

$$m = 1, 2, 3 \ldots..$$

ε = permittivity of plasma = $\varepsilon_o(1 - \omega_p^2/\omega^2)$

ε_o = permittivity of free space

ω_p^2 = (plasma frequency)^2 = $\dfrac{N_e e^2}{M\varepsilon_o}$

N_e = electron density

M = mass of electron

e = electronic charge

ω = probe signal angular frequency

c = speed of light in free space

a = width of rectangular guide

L = length of the plasma slab.

The number of the last peak shown in Fig. 14 is easily obtained since it corresponds to the shortening of the electrical length βL from its free space value to the closest odd multiple of $\pi/2$. A simple calculation gives n = 6 for this last peak.

A typical calculation from the data of Fig. 14 for the density at the peak reflection corresponding to n = 6 gives:

$$\omega_{p_6}^2 = \omega^2 - (13/2)^2 \frac{\pi^2 c^2}{L^2} - \frac{\pi^2 c^2}{a^2}$$

$$= 26.93 \times 10^{20} - 9.09 \times 10^{20} - 16.93 \times 10^{20}$$

$$= .91 \times 10^{20}$$

Since $\omega_p^2 = \dfrac{N_e e^2}{M\varepsilon_o} = 3.17 \times 10^9 N_e$ for the density N_e given in electrons per cubic centimeter, the value of N_{e6} is found to be

$$N_{e6} = 2.87 \times 10^{10} \text{ elec./cc.}$$

A plot of N_e versus time based on these calculations is given in Fig. 24. A calculation of the diffusion coefficient D can now be made based on the data of Fig. 24 and Eq. (2-37). The plasma density decays essentially as

$$N_e = N_{eo} \exp \{-[D_{xx}(\pi/a')^2 + D_{yy}(\pi/b')^2 + D_{zz}(\pi/L)^2]t\}$$

$$\doteq N_{eo} \exp \{-D[(\pi/a')^2 + (\pi/b')^2]t\} .$$

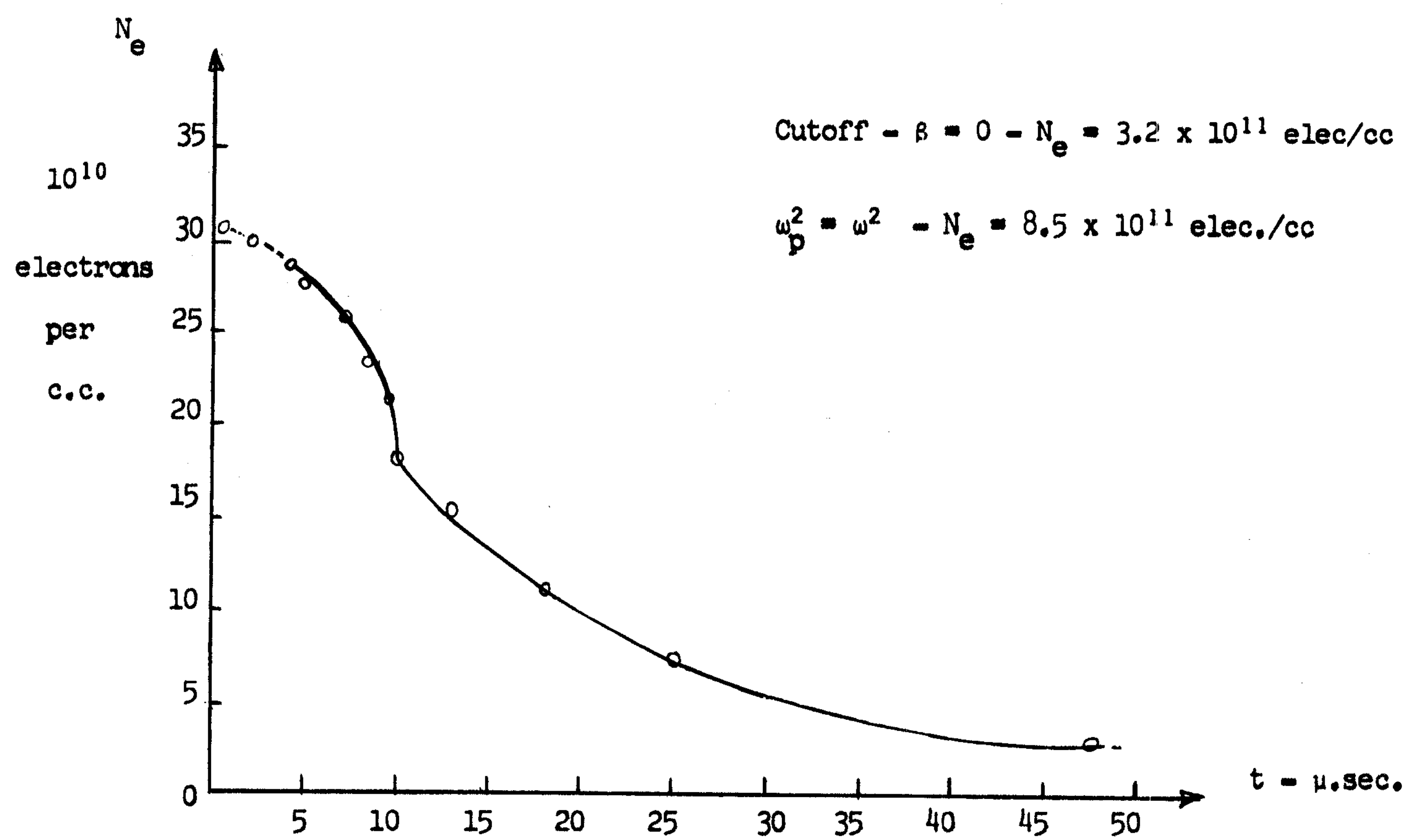
The value of D can be obtained by taking the derivative of $N_e(t)$ with respect to t and dividing by $N_e(t)$.

$$\frac{1}{N_e}\frac{dN_e}{dt} = -D[(\pi/a')^2 + (\pi/b')^2] = -D[22.7].$$

From the data of Fig. 24, ($t \gtrsim 40\mu$.sec.), the value of D is

$$D[22.7] = 4.2 \times 10^4$$

$$D = 1.85 \times 10^3 \text{ cm}^2/\text{sec.}$$



<u>Fig. 24</u> - N_e versus time for the data of Fig. 14.

REFERENCES

1. J. S. Townsend, Proc. Royal Society (London), A86, 571, (1912). Ibid, Electricity in Gases, Oxford University Press, Cambridge, England, 1915.

2. W. Schottky, Z. Physik, 25, 635, (1924).

3. V. E. Golant, Sov. Phys. Uspekhi, Russian Vol. 79, Nos. 3-4, pp. 161-197, Sept.-Oct., (1963).

4. F. C. Hoh, Revs. Mod. Phys., 34, 2, pp. 267-286, Apr., (1962).

5. H. J. Oskam, Phillips Research Report, 13, pp. 335-401, (1958).

6. S. Chapman and T. G. Cowling, The Mathematical Theory of Non-uniform Gases, Cambridge University Press, Cambridge, England, 1939.

7. J. L. Delcroix, Introduction to the Theory of Ionized Gases, Interscience Publishers, Inc., New York, 1960, pp. 75-80.

8. Lyman Spitzer, Jr., Physics of Fully Ionized Gases, Interscience Publishers, Inc., New York, 1956, pp. 94-98.

9. See, for example, V. E. Golant, Sov. Phys. Uspekhi, Russian Vol. 79, Nos. 3-4, pp. 162-164, Sept.-Oct., (1963).

10. E. P. Gray and D. E. Kerr, Annals of Physics, 17, pp. 276-300, (1959).

11. See, for example, R. W. Fredricks and F. Mastrup, Phys. Fluids, 6, 36, (1963).

12. H. B. G. Casmir, "Onsager's principle of microscopic reversibility," Revs. Modern Phys., 17, 343, (1945).

13. See, for example, M. A. Biondi and S. C. Brown, Phys. Rev., 75, 1700 (1949). Also, S. C. Brown, Basic Data of Plasma Physics, Wiley, New York, 1959.

14. K. G. Budden, Radio Waves in the Ionosphere, Cambridge University Press, Cambridge, England, 1961.

15. J. R. Wait, Electromagnetic Waves in Stratified Media, MacMillan Co., New York, 1962.

16. R. C. Cummings, "The Serrodyne frequency translator," Proc. IRE, 45, pp. 175-186, Feb., (1957).

17. C. A. Finnila, L. A. Roberts, and C. Susskind, "Measurement of relative phase shift at microwave frequencies," IRE, PGMTT, pp. 143-147, Mar., (1960).

18. J. B. Linker, Jr., and H. H. Grimm, "Automatic microwave transmission measuring equipment," Rev. Sci. Instr., 28, pp. 559-563, July, (1957).

19. See, R. E. Collin, Field Theory of Guided Waves, McGraw-Hill Co., New York, 1960.